

Functional Renormalisation Group analysis of Tensorial Group Field Theories on \mathbb{R}^d

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Rank- d Tensorial Group Field Theories are quantum field theories defined on a group manifold $G^{\times d}$, which represent a non-local generalisation of standard QFT, and a candidate formalism for quantum gravity, since, when endowed with appropriate data, they can be interpreted as defining a field theoretic description of the fundamental building blocks of quantum spacetime. Their renormalisation analysis is crucial both for establishing their consistency as quantum field theories, and for studying the emergence of continuum spacetime and geometry from them. In this paper, we study the renormalisation group flow of two simple classes of TGFTs, defined for the group $G = \mathbb{R}$ for arbitrary rank, both without and with gauge invariance conditions, by means of functional renormalisation group techniques. The issue of IR divergences is tackled by the definition of a proper thermodynamic limit for TGFTs. We map the phase diagram of such models, in a simple truncation, and identify both UV and IR fixed points of the RG flow. Encouragingly, for all the models we study, we find evidence for the existence of a phase transition of condensation type.

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I. INTRODUCTION

Group field theories [1] (GFTs) are a new type of quantum field theories characterised by a peculiar non-local pattern of pairings of field arguments in the interactions. The domain of definition of the fields is, for the most studied models, a (Lie) group manifold, hence the name of the formalism. The first consequence of the non-locality of the GFT interactions is that their quantum states can be associated to graphs (or networks), while the Feynman diagrams arising in the GFT perturbative expansion are dual to cellular complexes. These graphs and cellular complexes are then decorated by group-theoretic data, corresponding to the degrees of freedom associated to the GFT fields. This implies that a number of standard QFT techniques have to be adapted to this new context, and that a host of new mathematical structures can be explored by such field theoretic means. This formalism finds its historic roots, and main applications, at present, as a promising framework for quantum gravity. From this more physical perspective, GFTs are a tentative definition of the microstructure of quantum spacetime and of its fundamental quantum dynamics. The decorated graphs, in this interpretation, are the fundamental quantum structures from which a continuum spacetime and geometry should emerge in the appropriate regime of approximation. In fact, group field theories were first proposed [2] as an enrichment, by the addition of group-theoretic data, of tensor models [3] (in turn a generalisation of matrix models for 2d quantum gravity [4] to higher dimensions), with the main goal being to obtain Feynman amplitudes of the form of state sum models of topological field theories. The link with loop quantum gravity [5] became quickly clear [6]: group field theories and loop quantum gravity share the same type of quantum states, i.e. spin networks. It is then in the context of loop quantum gravity and state sum models, called spin foam models [7] and developed as a covariant definition of loop quantum gravity, that most subsequent work has been done, once it was understood [8] that the correspondence between group field theory and spin foam amplitudes is completely general. Finally, the relation between group field theory and lattice quantum gravity, already evident in their origin in tensor models, became stronger because of the appearance of the Regge action in semiclassical analyses of spin foam amplitudes (see for example, [9]), and, more recently, of the general possibility to recast group field theory amplitudes as (non-commutative) simplicial gravity path integrals [10]. It is now clear that group field theories sit at the crossroad of several approaches to quantum gravity, as a 2nd quantised framework for loop quantum gravity degrees of freedom [11] as well as an enrichment of tensor models. The quantum field theory framework they provide for the candidate fundamental degrees of freedom of quantum spacetime is then crucial for tackling the open issues of these approaches. In particular, it makes possible to take on them a condensed matter-like perspective, making precise the idea of ‘atoms of space’ and to study from this perspective the emergence of continuum spacetime [12], and to use powerful renormalisation group techniques to the analysis of their quantum dynamics. The renormalisation group analysis of GFT models has two main goals: establishing their perturbative renormalisability and exploring the continuum phase diagram. The first goal is all the more important because these models are initially defined and studied in perturbative expansion around the trivial vacuum, and it is in this expansion that their relation with loop quantum gravity and lattice quantum gravity, as well as their quantum geometric content, is more apparent. Establishing their perturbative renormalisability amounts then to establishing the consistency of this definition, and it also serves the purpose of constraining quantisation ambiguities (the GFT counterpart of those arising in the canonical loop quantum gravity formulation) as well as model building. The second goal is the most important open issue in all these related quantum gravity approaches: their continuum limit, i.e. the macroscopic, collective dynamics of their microscopic degrees of freedom, and the possibility of spacetime and geometry emerging from a phase transition of the same degrees of freedom [12], as it has been proposed also in related approaches [13–17]. It also amounts to controlling the full GFT expansion in terms of sum over cellular complexes and spin foam histories, thus it can be seen as solving, by QFT techniques, the problem of the continuum limit in both dynamical triangulations and spin foam models (for which alternative strategies are also been explored [18]).

GFT renormalisation is in fact one of the most rapidly developing research directions in this area, and it has benefited greatly from concurrent developments in tensor models [3], which provides analytic tools and many insights concerning the combinatorics and the topology of GFT Feynman diagrams [19, 20] as well as the possible definitions of the theory space to focus on [21, 22]. Indeed, most of the work in GFT renormalisation has concerned a class of GFTs, called

Tensorial Group Field Theories (TGFTs), in which tensorial structures are prominent. Several interesting TGFT models have been proven to be renormalisable [23, 24] and their RG flow has been also studied, mainly in the vicinity of the UV fixed point [25–28], showing that asymptotic freedom is a very general feature of TGFT models [29]. This work has encompassed abelian as well as non-abelian models, and both models with and without the additional gauge invariance properties that characterise GFTs for topological BF theory and 4d gravity, by giving their Feynman amplitudes the structure of lattice gauge theories. The same analysis has also been extended to models defined not on groups but on homogeneous spaces [30]. More recently, non-perturbative GFT renormalisation has been tackled as well. Some work [31, 32] has been based on the Polchinski equation and on the analysis of the Schwinger-Dyson equations (see also [33]). Most work has however been framed in the language of the Functional Renormalization Group approach to QFTs, first adapted to TGFTs in [34], after the initial steps taken in [35–38] for matrix models. The first model being studied [34] was an abelian rank-3 one on $U(1)$, and this analysis was quickly extended to the non-compact case in [39]. A model in rank-6 and again based on $U(1)$ was instead analysed in [40], this time incorporating gauge invariance. All these models were analysed in a fourth order truncation of in the number of fields. In all these cases, not only it was possible to confirm the asymptotic freedom of the models in the UV, but it was also possible to identify IR fixed points and to provide strong hints of a phase transition. The IR fixed points resemble Wilson-Fisher fixed points for ordinary scalar field theories, and the phase transition appears to separate a symmetric and a broken or condensate phase, with non-zero expectation value for the TGFT field operator. With a different perspective, the existence of phase transition has been proven for quartic tensor models in [41, 42] with a characterisation of the related phases and also for GFT models related to topological BF theory, in any dimension [43]. In models more directly related to loop quantum gravity and lattice quantum gravity, this type of phase transition was suggested to govern the emergence of an effective cosmological dynamics from such quantum gravity models [44].

In this paper, we generalise the analysis of abelian models on \mathbb{R} performed in [39], in two main ways: we compute and study the RG flow of models of arbitrary rank, and we perform the same analysis also for gauge invariant models, again in arbitrary rank. In both cases, we then specialise the results to rank 3, 4 and 5, identify the UV and IR fixed points and describe the resulting phase diagram. We still work, though, in a fourth order truncation of in the number of fields.

The plan of this paper is as follows. Section II reviews the Functional Renormalisation Group applied to Group Field Theories following [34]. In section III, we describe the analysis of the simplest class of non-compact models, without gauge invariance, for arbitrary rank. We also complete the analysis given in [39] by providing the solution of the system of β -function at second order around the Gaussian fixed point, provide details on the neighbourhood of that trivial fixed point. As in that previous work, the analysis of such non-compact model requires IR regularisation, as we will discuss in detail. The key point of our regularisation scheme is the introduction of a new parameter representing the dependence of couplings on the volume of the direct space. In the section IV, we repeat the analysis for another interesting class of models obtained introducing an additional gauge invariance in the amplitudes, by means of suitable projector operators inserted in the GFT action. After appropriate regularisation, we can again study the RG flow of these models. In section V we give a summary of our results and list some important open problems for this approach. Two appendices A and B provide more details of our calculations.

II. THE FUNCTIONAL RENORMALISATION GROUP FOR TGFTS: AN OVERVIEW

In this section we first review the basic ingredients of the (tensorial) group field theory formalism, in its covariant functional integral formulation. Then, we present the Functional Renormalisation approach, as it has been adapted and applied to TGFTs in [34].

A. Tensorial Group Field Theories

Let us introduce the special class of GFTs we will work with in the following, known as Tensorial Group Field Theories (TGFT) [23]–[28][45–47].

Consider a field ϕ defined over d -copies of a group manifold G , $\phi : G^{\times d} \rightarrow \mathbb{C}$. For the moment, we assume G to be a compact Lie group. Without assuming any symmetry under permutations of field labels and using Peter-Weyl theorem, the field decomposes in group representations as follows:

$$\phi(g_1, \dots, g_d) = \sum_{\mathbf{P}} \phi_{\mathbf{P}} \prod_{i=1}^d D^{p_i}(g_i), \quad (1)$$

with $\mathbf{P} = (p_1, \dots, p_d)$, $g_i \in G$ and where the functions $D^{p_i}(g_i)$ form a complete orthonormal basis of functions on the group characterised by the labels p_i . In a TGFT model, we require fields to have tensorial properties under basis

changes. We define a rank d covariant complex tensor $\phi_{\mathbf{P}}$ to transform through the action of the tensor product of unitary representations of the group $\bigotimes_{i=1}^d U^{(i)}$, each of them acting independently over the indices of field labels:

$$\phi_{p'_1, \dots, p'_d} = \sum_{\mathbf{P}} U_{p'_1, p_1}^{(1)} \cdots U_{p'_d, p_d}^{(d)} \phi_{p_1, \dots, p_d}. \quad (2)$$

The complex conjugate field will then be the contravariant tensor transforming as:

$$\bar{\phi}_{p'_1, \dots, p'_d} = \sum_{\mathbf{P}} (U^\dagger)_{p'_d, p_d}^{(d)} \cdots (U^\dagger)_{p'_1, p_1}^{(1)} \bar{\phi}_{p_1, \dots, p_d}. \quad (3)$$

TGFT interactions are defined by ‘trace invariants’ built out of ϕ and $\bar{\phi}$, which allow a strong control on the combinatorial structure of field convolutions, and are thus relevant for the construction of renormalisable TGFT actions. Tensorial trace invariants generalise invariant traces over matrices, which indeed are classical unitary invariants. They are obtained contracting pairwise the indices with the same position of covariant and contravariant tensors and saturating all of them. In this way, they always involve the same number of ϕ and $\bar{\phi}$. A simple example is the following:

$$\text{Tr}(\phi \bar{\phi}) = \sum_{\mathbf{P}, \mathbf{Q}} \phi_{\mathbf{P}} \bar{\phi}_{\mathbf{Q}} \prod_{i=1}^d \delta_{p_i, q_i}. \quad (4)$$

Considering that $\phi_{\mathbf{P}}$ (resp. $\bar{\phi}_{\mathbf{P}}$) transforms as a complex vector (resp. 1-form) under the action of the unitary representations of G on one single index, the fundamental theorem on classical invariants for U on each index entails that all invariant polynomials in field entries can be written as a linear combination of trace invariants [48]. This formulation of tensor models can be adapted to the real field case, where the unitary group is replaced by the orthogonal one [49].

An interesting feature, which becomes an important computational tool, is that tensor invariants can be given a graphical representation as bipartite coloured graphs, and in fact they are in one to one correspondence with them. A tensor ϕ is represented by a (white) node with d labelled half lines outgoing from it. Its complex conjugate is a similar d -valent node with a different colour (black). A tensor contraction is represented then by joining the half-lines, equally labelled, of two nodes of different colour.

Trace invariants can be generalised to convolutions where the contractions are made by operators different from the delta distribution, i.e. by non-trivial kernels. In this case, the resulting object is not guaranteed to be a unitary invariant.

We write a generic action for a TGFT model symbolically as:

$$\begin{aligned} S[\phi, \bar{\phi}] &= \text{Tr}(\bar{\phi} \cdot \mathcal{K} \cdot \phi) + S^{\text{int}}[\phi, \bar{\phi}] \\ \text{Tr}(\bar{\phi} \cdot \mathcal{K} \cdot \phi) &= \sum_{\mathbf{P}, \mathbf{Q}} \bar{\phi}_{\mathbf{P}} \mathcal{K}(\mathbf{P}; \mathbf{Q}) \phi_{\mathbf{Q}}, \quad S^{\text{int}}[\phi, \bar{\phi}] = \sum_{\{n_b\}} \lambda_{n_b} \text{Tr}(\mathcal{V}_{n_b} \cdot \phi^n \cdot \bar{\phi}^n). \end{aligned} \quad (5)$$

Here \mathcal{K} and \mathcal{V}_n are kernels implementing the convolutions in the kinetic and interaction terms, respectively, where n indicates the numbers of covariant and contravariant fields appearing in the vertices, b labels the combinatorics of convolutions (i.e. corresponds to some given bipartite d -coloured graph) and λ_{n_b} is a coupling constant for the interaction n_b .

The formalism can be easily generalised to a TGFT based on a non-compact group manifold G , and in this case the Plancherel decomposition into (unitary) representations replaces the Peter-Weyl one to decompose fields, and the definition of the trace over representation labels involves, in general, also integrals over continuous variables.

Given an action $S[\phi, \bar{\phi}]$, the partition function is defined as usual:

$$\mathcal{Z}[J, \bar{J}] = e^{W[J, \bar{J}]} = \int d\phi d\bar{\phi} e^{-S[\phi, \bar{\phi}] + \text{Tr}(J \cdot \bar{\phi}) + \text{Tr}(\bar{J} \cdot \phi)}, \quad (6)$$

where J is a rank d complex source term and $\text{Tr}(J \cdot \bar{\phi})$ is defined in (4).

The partition function can be expanded in perturbation theory around a Gaussian distribution, and expressed as a (formal) sum over Feynman diagrams. Feynman diagrams of a rank- d TGFT are obtained by attaching, to the bipartite graph corresponding to a trace invariant defining each interaction vertex, a propagator (dashed line) for each field obtaining a $(d+1)$ -coloured graph (some examples are depicted in Fig.1).

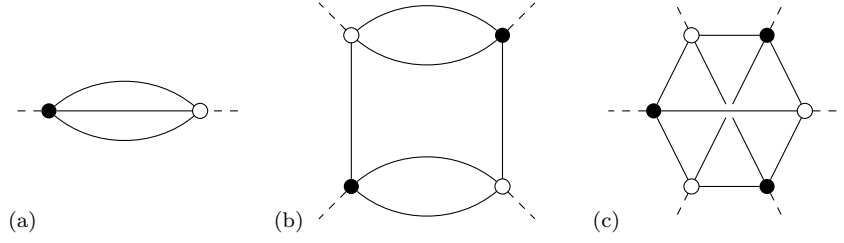


FIG. 1. Three examples of Feynman graphs for a rank 3 TGFT's. The trace invariants used to build the interactions are: figure (a) $\text{Tr}(\phi\bar{\phi})$, figure (b) an example of $\text{Tr}(\phi\bar{\phi}\phi\bar{\phi})$, figure (c) an example of $\text{Tr}(\phi\bar{\phi}\phi\bar{\phi}\phi\bar{\phi})$.

B. FRG formulation for TGFTs

The generalisation of the FRG formalism [50–54] to TGFTs is straightforward and was first provided in [34]. Given a partition function of the type (6), we choose a UV cut-off M and a IR cut-off N^1 . Adding to the action a regulator term of the form:

$$\Delta S_N[\phi, \bar{\phi}] = \text{Tr}(\bar{\phi} \cdot R_N \cdot \phi) = \sum_{\mathbf{P}, \mathbf{P}'} \bar{\phi}_{\mathbf{P}} R_N(\mathbf{P}; \mathbf{P}') \phi_{\mathbf{P}'} , \quad (7)$$

we can perform the usual splitting in high and low modes. In particular, given an action with a generic kernel depending on the derivative of the fields $\mathcal{K}(\nabla\phi)$ and a generalised Fourier transform \mathcal{F} , if we choose R_N to be of the specific form

$$R_N(\mathbf{P}; \mathbf{P}') = N \delta_{\mathbf{P}, \mathbf{P}'} R\left(\frac{\mathcal{F}(\mathcal{K}_{\mathbf{P}})}{N}\right) , \quad (8)$$

we need to impose on the profile function $R(z)$ the following conditions:

- positivity $R(z) \geq 0$, to indeed suppress and not enhance modes outside of the domain of the regulator function;
- monotonicity $\frac{d}{dz}R(z) \leq 0$, so that high modes will not be suppressed more than low modes;
- $R(0) > 0$ and $\lim_{z \rightarrow +\infty} R(z) = 0$ to exclude functions with constant profile.

The last requirement, together with the form (8), guarantees that the regulator is removed for $Z \rightarrow 0$. In accordance with the usual FRG procedure, we define the scale dependent partition function as:

$$\mathcal{Z}_N[J, \bar{J}] = e^{W_N[J, \bar{J}]} = \int d\phi d\bar{\phi} e^{-S[\phi, \bar{\phi}] - \Delta S_N[\phi, \bar{\phi}] + \text{Tr}(J \cdot \bar{\phi}) + \text{Tr}(\bar{J} \cdot \phi)} \quad (9)$$

and the generating functional of 1PI correlation functions after Legendre transform are given in terms of the average field $\varphi = \langle \phi \rangle$ as

$$\Gamma_N[\varphi, \bar{\varphi}] = \sup_{J, \bar{J}} \left\{ \text{Tr}(J \cdot \bar{\varphi}) + \text{Tr}(\bar{J} \cdot \varphi) - W_N[J, \bar{J}] - \Delta S_N[\varphi, \bar{\varphi}] \right\} . \quad (10)$$

Given the above definitions, the Wetterich equation takes the form:

$$\partial_t \Gamma_N[\varphi, \bar{\varphi}] = \overline{\text{Tr}} \left(\partial_t R_N \cdot [\Gamma_N^{(2)} + R_N]^{-1} \right) , \quad (11)$$

where $t = \log N$, so that $\partial_t = N \partial_N$, and the “super”-trace symbol $\overline{\text{Tr}}$ means that we are summing over all mode labels. More explicitly, the functional trace reads:

$$\sum_{\mathbf{P}, \mathbf{P}'} \partial_t R_N(\mathbf{P}; \mathbf{P}') [\Gamma_N^{(2)} + R_N]^{-1}(\mathbf{P}'; \mathbf{P}) . \quad (12)$$

The presence of the $\partial_t R_N$ in the Wetterich equation for TGFT's, enforces the trace to be UV-finite if the profile function and its derivative go fast enough to 0, as $z \rightarrow +\infty$. In this way, we can basically forget about the UV cut-off M . In any case, as in any resolution of differential equation, we need an initial condition of the type

$$\Gamma_{N=M}[\varphi, \bar{\varphi}] = S[\varphi, \bar{\varphi}] , \quad (13)$$

¹ We adopt a standard QFT terminology for field modes, even if no spacetime interpretation should be attached to them, at this stage.

for some scale M . The problem of solving the full quantum theory is now phrased in the one of pushing the initial condition to infinity, which usually requires the existence of a UV fixed point, and solving the Wetterich equation with such initial condition. The full quantum field theory will then be defined by the corresponding solution, i.e. by the full RG trajectory.

The Wetterich equation has a 1-loop structure, and since no (perturbative) approximation is required to obtain it, it is an exact functional equation. However, although we have expressed the problem of extracting the flow of the theory in terms of a partial differential equation in one single variable, we still have the issue that all possible (i.e. compatible with symmetry requirements and field content) couplings are allowed in Γ_k , which is thus expressible as an infinite sum of monomials in the field (and its conjugate). If we want to perform practical computations, we need some approximation scheme for the form of the free energy. Usually, this is obtained by truncating Γ_k to a maximal power in the fields and in their derivative. It is then a truncation in theory space, which maintains the non-perturbative character of the RG equation.

What is peculiar, and interesting, about the application of FRG to TGFTs, is that $\Gamma_N^{(2)}$ carries inside the Wetterich equation information about the combinatorial non-locality of the theory, i.e. the intricate combinatorics of TGFT interactions. In the case we consider here, that of a non-compact group manifold, this will also back-react at the level of the β -functions, in the fact that, depending on the combinatorics of the interaction, the volume contributions appearing in (11) will be not homogeneous and, in general, a natural definition of an effective local potential does not exist. Let us explain this key point, which we will deal with in detail in the following.

In its usual form, namely when applied to a standard, local quantum field theory (see for instance, in [52]), the Wetterich equation shows pathological IR divergences due to the presence of $\delta(0)$ arising from the two-point Green's function computed at a single point $G_k^{(2)}(q, q)$. In the local field theory case, these divergent delta functions are homogeneous and proportional to the total volume of the system, namely, the domain manifold of the fields. A particular approximation procedure allows to cure this problem and it is called the local potential approximation (LPA) [52]. This procedure cannot be applied, at least not in the same straightforward way, to combinatorially non-local theories as TGFTs. One reason is that, in such non-local theories, the same type of IR divergence arise, in general, in a non-homogeneous combination of $\delta(0)$ which are strictly dependent on the combinatorics of the interaction. We will discuss this and several other issues characterising TGFTs as QFTs of an interesting new kind.

III. RANK- d TENSORIAL GROUP FIELD THEORY ON \mathbb{R}

As discussed in the introduction, the first model studied within the FRG framework for TGFTs, already in [34], was a rank-3 model with compact group manifold $U(1)$, and subsequently, we have studied a non-compact counterpart of the same model, i.e. a rank-3 TGFT on \mathbb{R} [39]. New issues concerning the thermodynamic limit but also more compelling hints for the existence of UV and IR fixed points, and of a condensation phase transitions, were found.

We now extend the analysis and results of the latter work to arbitrary rank (as well as analysing in more detail in the rank-3 model), showing how those intriguing hints are actually confirmed in a more general case. In the following section, we will analyse a modification of the same type of TGFT models which includes a gauge invariance property of fields and amplitudes, thus moving closer to full-fledged TGFT models for quantum geometry and discrete quantum gravity, and related to loop quantum gravity.

We start by introducing the class of TGFT models we will analyse.

A. The model

The TGFTs we work with have “melonic” interactions (in correspondence with d -colored graphs called “melons”) [55–57]. Such melons are dual to special triangulations of the d -ball [47] and of course correspond also to trace invariants of the type introduced in section II A.

We consider a rank- d model with complex field, $\phi : \mathbb{R}^d \rightarrow \mathbb{C}$, defined by the following action:

$$\begin{aligned}
S[\phi, \bar{\phi}] = & (2\pi)^d \int_{\mathbb{R} \times d} [dx_i]_{i=1}^d \bar{\phi}(x_1, \dots, x_d) \left(- \sum_{s=1}^d \Delta_s + \mu \right) \phi(x_1, \dots, x_d) \\
& + \frac{\lambda}{2} (2\pi)^{2d} \int_{\mathbb{R} \times 2d} [dx_i]_{i=1}^d [dx'_j]_{j=1}^d \left[\phi(x_1, x_2, \dots, x_d) \bar{\phi}(x'_1, x_2, \dots, x_d) \phi(x'_1, x'_2, \dots, x'_d) \bar{\phi}(x_1, x'_2, \dots, x'_d) \right. \\
& \left. + \text{sym}\{1, 2, \dots, d\} \right],
\end{aligned} \tag{14}$$

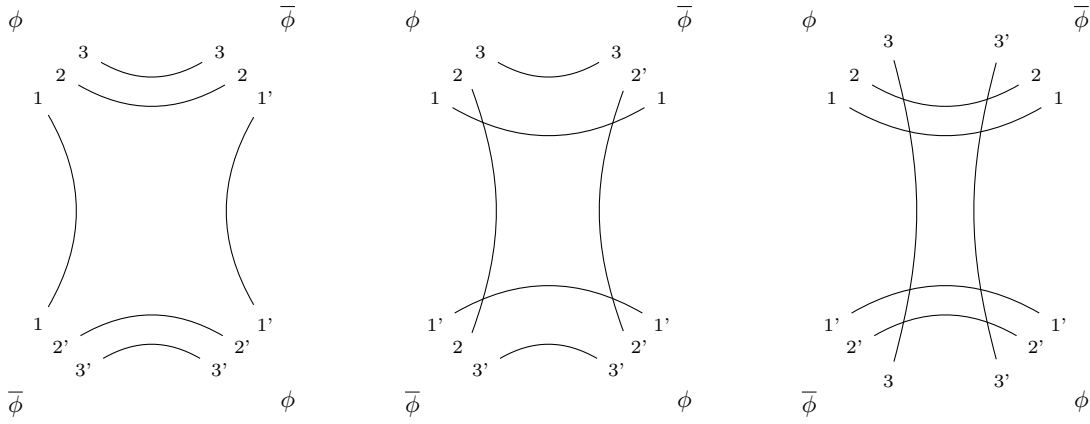


FIG. 2. Colored symmetric interaction terms in rank $d = 3$.

where 2π factors have been conveniently introduced in the definition of the Fourier transform, the symbol $\text{sym}\{\cdot\}$ represents the rest of the coloured symmetric terms in the interaction (see Fig.2 for a graphical representation² in rank $d = 3$); μ and λ are coupling constants. As it is easy to see, due to the structure of the interaction kernels, the interaction fully depends on all the six coordinates and this makes it non-local from the combinatorial point of view.

After Fourier transform, we write the action in momentum space as:

$$S[\phi, \bar{\phi}] = \int_{\mathbb{R} \times d} [dp_i]_{i=1}^d \bar{\phi}_{12\dots d} \left(\sum_{s=1}^d p_s^2 + \mu \right) \phi_{12\dots d} \quad (15)$$

$$+ \frac{\lambda}{2} \int_{\mathbb{R} \times 2d} [dp_i]_{i=1}^d [dp'_j]_{j=1}^d \left[\phi_{12\dots d} \bar{\phi}_{1'2'\dots d} \phi_{1'2'\dots d'} \bar{\phi}_{12'\dots d'} + \text{sym}\{1, 2, \dots, d\} \right],$$

where we use the conventions

$$\phi_{12\dots d} = \phi_{p_1, p_2, \dots, p_d} = \phi(\mathbf{p}) = \int_{\mathbb{R} \times d} [dx_i]_{i=1}^d \phi(x_1, x_2, \dots, x_d) e^{-i \sum_i p_i x_i}, \quad (16)$$

$$\phi(x_1, x_2, \dots, x_d) = \frac{1}{(2\pi)^d} \int_{\mathbb{R} \times d} [dp_i]_{i=1}^d \phi_{12\dots d} e^{i \sum_i p_i x_i}. \quad (17)$$

We represent the propagator as a stranded line made with d segments (strands). See Fig.3 for the case $d = 3$. The combinatorics of the interaction is preserved by the Fourier transform.

$$\bar{\phi} \quad \text{=====} \quad \phi \quad = \left(\sum_s p_s^2 + \mu \right)^{-1}$$

FIG. 3. Feynman rule for the propagator at $d = 3$.

We can now proceed with the dimensional analysis to fix the dimensions of the coupling constants. In order to make sense of the exponentiation of the action in the partition function, we must set $[S] = 0$. Furthermore, we fix the dimensions to be in unit of the momentum, i.e., $[p] = [dp] = 1$ ³. Now, for consistency we must have $[\mu] = 2$. This leads us to the following equations:

$$3 + 2[\phi] + 2 = 0 \Rightarrow [\phi] = -\frac{d+2}{2}, \quad (18)$$

$$[\lambda] + 2d + 4[\phi] = 0 \Rightarrow [\lambda] = 4, \quad (19)$$

which fix the dimension of the TGFT fields depending on the rank d of the model.

² As a remark, in the following subsections, illustrations and figures are made in the case $d = 3$ because the general case can be easily recovered from that case.

³ Notice that the physical dimension of such momentum variables, if any, is not especially relevant in this context; what matters is the relative dimension of the various ingredients entering the TGFT action.

B. Effective action and Wetterich equation

In order to proceed with the Functional Renormalisation Group analysis, following the general template described in the previous section, we introduce an IR cut-off k and a UV cut-off Λ . We need to perform a truncation on the form of the effective action. A natural choice, compatible with the condition (13), is to truncate the effective action to be of the same form of the action itself for any value of the cut-offs, that is:

$$\begin{aligned} \Gamma_k[\varphi, \bar{\varphi}] = & \int_{\mathbb{R} \times d} [dp_i]_{i=1}^d \bar{\varphi}_{12\dots d} (Z_k \sum_s p_s^2 + \mu_k) \varphi_{12\dots d} \\ & + \frac{\lambda_k}{2} \int_{\mathbb{R} \times 2d} [dp_i]_{i=1}^d [dp'_j]_{j=1}^d \left[\varphi_{12\dots d} \bar{\varphi}_{1'2'\dots d'} \varphi_{1'2'\dots d'} \bar{\varphi}_{12\dots d} + \text{sym}\{1, 2, \dots, d\} \right], \end{aligned} \quad (20)$$

where $\varphi = \langle \phi \rangle$. As we have already stressed, this is a non-perturbative truncation of the theory, and any of the ensuing results should then be tested by extending this truncation, including more invariants (including other types of $\text{Tr}(\phi^4)$ invariants, i.e. with different combinatorics, as well as higher order terms $\text{Tr}(\phi^{2n})$, $n \geq 3$; in general, one should include also disconnected invariants such as multi-traces, $\text{Tr}(\phi^{2n})\text{Tr}(\phi^{2m})\dots$) and checking for (qualitative) convergence. Enlarging the theory space is postponed for future investigations, but it should be obvious that, even in the truncation given by (20), the calculations and the outcome of the present analysis remain highly non-trivial.

From the dimensional analysis of the previous section and from the fact that $[\Gamma_k] = 0$ and $[\varphi] = [\phi]$, one infers $[Z_k] = 0$, $[\mu_k] = [\mu] = 2$, $[\lambda_k] = [\lambda] = 4$.

We introduce a regulator kernel of the following form [58, 59]

$$R_k(\mathbf{p}, \mathbf{p}') = \delta(\mathbf{p} - \mathbf{p}') Z_k (k^2 - \sum_s p_s^2) \theta(k^2 - \sum_s p_s^2), \quad (21)$$

where θ stands for the Heaviside step function. This form of the regulator is convenient because it allows to solve analytically many spectral sums. It is easy to show that R_k satisfies the minimal requirements for a regulator kernel:

- as a consequence of the fact that $\theta(-|x|) = 0$, we have

$$R_{k=0}(\mathbf{p}, \mathbf{p}') = \delta(\mathbf{p} - \mathbf{p}') Z_k (-\sum_s p_s^2) \theta(-\sum_s p_s^2) = 0; \quad (22)$$

- at the scale $k = \Lambda$, the regulator takes the form:

$$R_{k=\Lambda}(\mathbf{p}, \mathbf{p}') = \delta(\mathbf{p} - \mathbf{p}') Z_\Lambda (\Lambda^2 - \sum_s p_s^2) \theta(\Lambda^2 - \sum_s p_s^2), \quad (23)$$

which at the first order gives: $R_{k=\Lambda} \simeq Z_\Lambda \Lambda^2$;

- for $k \in [0, \Lambda]$, we have also:

$$R_k(\mathbf{p}, \mathbf{p}') = 0, \quad \forall \mathbf{p}, \mathbf{p}', \text{ such that } |\mathbf{p}|, |\mathbf{p}'| > k, \quad (24)$$

$$R_k(\mathbf{p}, \mathbf{p}') \simeq Z_k k^2, \quad \forall \mathbf{p}, \mathbf{p}', \text{ such that } |\mathbf{p}|, |\mathbf{p}'| < k. \quad (25)$$

The derivative of the regulator kernel with respect to the logarithmic scale $t = \log k$, entering in the Wetterich equation, evaluates as:

$$\partial_t R_k(\mathbf{p}, \mathbf{p}') = \theta(k^2 - \sum_s p_s^2) [\partial_t Z_k (k^2 - \sum_s p_s^2) + 2k^2 Z_k] \delta(\mathbf{p} - \mathbf{p}'). \quad (26)$$

One notes that R_k and $\partial_t R_k$ are both symmetric kernels, which is important in evaluating the convolutions induced by the Wetterich equation.

Computing the 1PI 2-point function yields:

$$\begin{aligned} \Gamma_k^{(2)}(\mathbf{q}, \mathbf{q}') = & (Z_k \sum_s q_s^2 + \mu_k) \delta(\mathbf{q} - \mathbf{q}') + \lambda_k \left[\int_{\mathbb{R}} dp_1 \varphi_{p_1 q'_2 \dots q'_d} \bar{\varphi}_{p_1 q_2 \dots q_d} \delta(q_1 - q'_1) \right. \\ & + \left. \int_{\mathbb{R} \times (d-1)} [dp_i]_{i=2}^d \varphi_{q'_1 p_2 \dots p_d} \bar{\varphi}_{q_1 p_2 \dots p_d} \left[\prod_{i=2}^d \delta(q_i - q'_i) \right] + \text{sym}\{1, 2, \dots, d\} \right] \\ = & (Z_k \sum_s q_s^2 + \mu_k) \delta(\mathbf{q} - \mathbf{q}') + F_k(\mathbf{q}, \mathbf{q}'). \end{aligned} \quad (27)$$

There is a simple graphical way to picture the various terms contributing to F_k . Each summed index can be represented by a segment and each fixed index (not summed) by a dot. As an example in rank $d = 3$, Fig.4 displays two terms coming from the second variation of the interaction labeled by colour 1 (the ones which appear explicitly in (27)). The other terms appearing in $\text{sym}\{\cdot\}$ can be inferred by colour permutation.



FIG. 4. Terms of the second variation of Γ_k at rank $d = 3$.

Defining the operator P_k with kernel

$$P_k(\mathbf{p}, \mathbf{p}') = R_k(\mathbf{p}, \mathbf{p}') + (Z_k \sum_s p_s^2 + \mu_k) \delta(\mathbf{p} - \mathbf{p}'), \quad (28)$$

the Wetterich equation can be recast as:

$$\partial_t \Gamma_k = \text{Tr}[\partial_t R_k \cdot (P_k + F_k)^{-1}]. \quad (29)$$

The r.h.s. of (29) generates an infinite series of terms with convolutions involving an arbitrary number of fields. In order to compare the two sides of (29), we must therefore perform a truncation in this series to match with the l.h.s. of that equation. This may be achieved expanding (29) in powers of $F_k \cdot (P_k)^{-1}$, that is, in powers of $\varphi\bar{\varphi}$, and considering only the terms up to the power 2:

$$\begin{aligned} \partial_t \Gamma_k &= \text{Tr}[\partial_t R_k \cdot (P_k)^{-1} \cdot (1 + F_k \cdot (P_k)^{-1})^{-1}] \\ &= \text{Tr}[\partial_t R_k \cdot (P_k)^{-1} \cdot (1 - F_k \cdot (P_k)^{-1} + F_k \cdot (P_k)^{-1} \cdot F_k \cdot (P_k)^{-1} + o((\varphi\bar{\varphi})^3))]. \end{aligned} \quad (30)$$

The vacuum term proportional to the 0-th order in the above expansion will be discarded because it does not reflect any term in the l.h.s. of (29). As an explicit example, the trace at linear order takes the form:

$$\partial_t \Gamma_k^{\text{kin}} = \int_{\mathbb{R} \times 12} \partial_t R_k(\mathbf{p}, \mathbf{p}') (P_k)^{-1}(\mathbf{p}', \mathbf{q}) F_k(\mathbf{q}, \mathbf{q}') (P_k)^{-1}(\mathbf{q}', \mathbf{p}). \quad (31)$$

Already, from the structure of the operators, $\partial_t R_k$, P_k and F_k , we expect the presence of singular $\delta(0)$ -terms which need to be regularised. Indeed, the appearance of $\delta(0)$ -terms reflects the fact that we have infinite volume effects which have to be treated. The presence of such infinities, as we have anticipated above, is not a specific feature of TGFTs, as it also arises in standard QFT. What is peculiar in TGFTs is the fact that, due to the combinatorics of the vertex operators, these divergences cannot be addressed by projection on the constant fields. Roughly speaking, in ordinary (local) field theories projecting on constant fields allows to factorise out the full volume of the space entering some given power of $\delta(0)$, and depending only on the order of the field interaction. Such a procedure cannot be applied in the present setting, the main reason being that the order of volume divergences depends not only on the order of field interactions but also on their precise convolution pattern. This would be entirely lost in a constant field projection, and must instead be checked term-by-term in the expansion of (29). The best way to tackle these divergences is to resort to a compactification of configuration space, corresponding to a discretisation in the conjugate space, and define an appropriate thermodynamic limit. This is explained in the next section.

C. IR divergences and thermodynamic limit

In order to regularise volume divergences, we perform a compactification of the direct space and a lattice regularisation in the conjugate space, following the conventions of [60], and generalising to arbitrary rank the procedure adopted in [39]. Defining the model (14) over a compact set $D \subset \mathbb{R}^{\times d}$ with volume $L^d = (2\pi r)^d$, and taking a Fourier transform, the domain of integration (actually, summation) of the effective action, in momentum space, becomes the lattice

$$D^* = \left(\frac{2\pi}{L} \mathbb{Z}\right)^{\times d} = \left(\frac{1}{r} \mathbb{Z}\right)^{\times d} := (l\mathbb{Z})^{\times d}, \quad (32)$$

so that we have, for any function $F(\mathbf{p})$,

$$\int_{D^*} [dp_i]_{i=1}^d F(\mathbf{p}) = l^d \sum_{p_1, p_2, \dots, p_d \in D^*} F(\mathbf{p}). \quad (33)$$

We define the delta distribution in D^* in momentum space as:

$$\delta_{D^*}(\mathbf{p}, \mathbf{q}) = l^{-d} \delta_{\mathbf{p}, \mathbf{q}}. \quad (34)$$

with $\delta_{\mathbf{p}, \mathbf{q}} = \prod_s \delta_{p_s, q_s}$, the Kronecker delta. Choosing an orthonormal basis $(e_{\mathbf{p}})_{\mathbf{p} \in D^*}$ for the space of fields such that $e_{\mathbf{p}}(\mathbf{q}) = \delta_{D^*}(\mathbf{p}, \mathbf{q})$, we have:

$$\phi(\mathbf{p}) = \langle e_{\mathbf{p}}, \phi \rangle_{D^*}. \quad (35)$$

For a generic observable A , we then have

$$(A\phi)(\mathbf{p}) = \int_{D^*} [dq_i]_{i=1}^d A(\mathbf{q}, \mathbf{p}) \phi(\mathbf{p}) = \int_{D^*} [dq_i]_{i=1}^d \langle e_{\mathbf{q}}, A e_{\mathbf{p}} \rangle_{D^*} \phi(\mathbf{p}). \quad (36)$$

Whenever A is invertible, then the inverse operator satisfies

$$\int_{D^*} [dr_i]_{i=1}^d A(\mathbf{p}, \mathbf{r}) A^{-1}(\mathbf{r}, \mathbf{q}) = \delta_{D^*}(\mathbf{p}, \mathbf{q}). \quad (37)$$

We also define the regularised functional derivative as:

$$\frac{\delta}{\delta \phi(\mathbf{p})} = l^{-d} \frac{\partial}{\partial \phi(\mathbf{p})}, \quad (38)$$

so that the following relations hold:

$$\frac{\delta}{\delta \phi(\mathbf{p})} \phi(\mathbf{q}) = \delta_{D^*}(\mathbf{p}, \mathbf{q}), \quad \frac{\delta}{\delta J(\mathbf{p})} e^{\langle J, \phi \rangle_{D^*}} = J(\mathbf{p}) e^{\langle J, \phi \rangle_{D^*}}. \quad (39)$$

This set of conventions is of course consistent with the continuous version of field theory, where δ_{D^*} becomes the Dirac δ -distribution and the derivative (38) becomes the standard functional derivative.

Using this regularization prescription, the effective action of the model takes the form:

$$\begin{aligned} \Gamma_k[\varphi, \bar{\varphi}; l] = & l^d \sum_{\mathbf{p} \in D^*} \bar{\varphi}_{12\dots d} \left(Z_k \sum_s p_s^2 + \mu_k \right) \varphi_{12\dots d} \\ & + l^{2d} \frac{\lambda_k}{2} \sum_{\mathbf{p}, \mathbf{p}' \in D^*} \left[\varphi_{12\dots d} \bar{\varphi}_{1'2'\dots d'} \varphi_{1'2'\dots d'} \bar{\varphi}_{12\dots d} + \text{sym}\{1, 2, \dots, d\} \right], \end{aligned} \quad (40)$$

where, using the same notation φ for the field and its Fourier transform, one has:

$$\begin{aligned} \varphi(x_1, x_2, \dots, x_d) &= (2\pi)^{-d} l^d \sum_{\mathbf{p} \in D^*} e^{i \sum_i p_i x_i} \varphi(\mathbf{p}), \\ \varphi(\mathbf{p}) &= \int_D [dx_i]_{i=1}^d e^{-i \sum_i p_i x_i} \varphi(x_1, x_2, \dots, x_d). \end{aligned} \quad (41)$$

Now we use the relations (41) to transform δ_{D^*} and obtain

$$(2\pi)^{-d} l^d \sum_{\mathbf{p} \in D^*} \delta_{D^*}(\mathbf{p}, \mathbf{q}) e^{i \sum_i p_i x_i} = (2\pi)^{-d} e^{i \sum_i q_i x_i}. \quad (42)$$

Thus, an integral representation of the delta distribution over D^* can be consistently defined as

$$\delta_{D^*}(\mathbf{p}, \mathbf{q}) = (2\pi)^{-d} \int_D [dx_i]_{i=1}^d e^{-i \sum_i (p_i - q_i) x_i}. \quad (43)$$

As a final result, we have:

$$\delta_{D^*}(\mathbf{p}, \mathbf{p}) = \frac{(2\pi r)^d}{(2\pi)^d} = r^d = \frac{1}{l^d}. \quad (44)$$

From these formulae, the continuum description will be recovered in the thermodynamic limit $l \rightarrow 0$.

This procedure makes the dependence on the volume of the direct space explicit. We can then rescale also the coupling constants of the model so to incorporate in their definition a dependence on the same volume. Then, we can use this dependence in such a way that the non-compact (thermodynamic) limit of the theory becomes well defined and all divergences are consistently removed.

D. β -functions and RG flows

We introduce a regularisation as outlined in section III C, and write the regularised effective action as:

$$\Gamma_k[\varphi, \bar{\varphi}] = \int_{D^*} [dp_i]_{i=1}^d \bar{\varphi}_{12\dots d} (Z_k \sum_s p_s^2 + \mu_k) \varphi_{12\dots d} + \frac{\lambda_k}{2} \int_{D^* \times 2} [dp_i]_{i=1}^d [dp'_j]_{j=1}^d \left[\varphi_{12\dots d} \bar{\varphi}_{1'2'\dots d'} \varphi_{1'2'\dots d'} \bar{\varphi}_{12\dots d} + \text{sym}\{1, 2, \dots, d\} \right]. \quad (45)$$

We can study the Wetterich equation corresponding to the action (45), incorporating a dependence on the volume in the coupling constants, and perform a thermodynamic limit at the end of the computation to extract the coefficients valid in the non-compact case.

The set of β -functions that we obtain from the discretised model is⁴:

$$\left\{ \begin{aligned} \beta(Z_k) &= \frac{\lambda_k}{(Z_k k^2 + \mu_k)^2} \left\{ \partial_t Z_k \left[2(d-1) \frac{k}{l} + \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+1}{2}\right)} \frac{k^{d-1}}{l^{d-1}} \right] + 2Z_k \left[(d-1) \frac{k}{l} + \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d-1}{2}\right)} \frac{k^{d-1}}{l^{d-1}} \right] \right\} \\ \beta(\mu_k) &= -\frac{d\lambda_k}{(Z_k k^2 + \mu_k)^2} \left\{ \partial_t Z_k \left[\frac{4}{3} \frac{k^3}{l} + \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+3}{2}\right)} \frac{k^{d+1}}{l^{d-1}} \right] + 2Z_k \left[2 \frac{k^3}{l} + \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+1}{2}\right)} \frac{k^{d+1}}{l^{d-1}} \right] \right\} \\ \beta(\lambda_k) &= \frac{2\lambda_k^2}{(Z_k k^2 + \mu_k)^3} \left\{ \partial_t Z_k \left[\frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+3}{2}\right)} \frac{k^{d+1}}{l^{d-1}} + \frac{4(2d-1)}{3} \frac{k^3}{l} + 2\delta_{d,3} k^2 \right] \right. \\ &\quad \left. + 2Z_k \left[\frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+1}{2}\right)} \frac{k^{d+1}}{l^{d-1}} + 2(2d-1) \frac{k^3}{l} + 2\delta_{d,3} k^2 \right] \right\} \end{aligned} \right. \quad (46)$$

It must be stressed that the coefficients appearing in (46) are computed with integrals like in the continuous setup. This is however not an issue, once the volume dependence has been factored out, the order of taking the limit and performing the integral does not matter.

Some interesting features of the system (46) must be stressed. At this intermediate step (the limit $\lim_{l \rightarrow 0}$ still has to be taken), this is a non-autonomous system and it involves terms of different powers in the cut-off k (we refer to this feature as “non-homogeneity” in k). Non-autonomous systems are known to occur in other contexts, for example quantum field theory at finite temperature [54], or on a curved [61] and non-commutative spacetime [62]. The non-homogeneity in k of the system signals the presence of an external scale, for the system; here, the radius of the compactified configuration space. The specific form of the terms appearing in this case is an effect of the particular combinatorics of the vertices of the theory which, after differentiation, yields 1PI 2-point function with terms with different volume contributions. If the l parameter is kept finite, we see two different system arising in the UV and IR limits, coming from different leading terms. Such a feature has been found in previous work [34] and both the two limits and the intermediate regime investigated. In the two limits one can compute the analogue of fixed points, which however cannot be straightforwardly interpreted as such.

On the other hand, if one tries to proceed in the usual way, extracting the dimensions of the coupling constants using one parameter (k or l), one obtains a set of β -functions which are either trivial or still divergent in the limit. Hence, in the end the non-local combinatorics of the TGFT interactions requires a drastic revision of conventional procedures of local QFTs. As we now show the correct way of proceeding in the TGFT case requires taking advantage of the presence of both the two parameters (k, l), when defining the scaling of the couplings.

To make sense of the above system, consider the following ansatz:

$$Z_k = \bar{Z}_k l^\chi k^{-\chi}, \quad \mu_k = \bar{\mu}_k \bar{Z}_k l^\chi k^{2-\chi}, \quad \lambda_k = \bar{\lambda}_k \bar{Z}_k^2 l^\xi k^\sigma, \quad (47)$$

where $[\bar{Z}_k] = [\bar{\mu}_k] = [\bar{\lambda}_k] = 0$, $[\varphi] = -\frac{d+2}{2}$ and $\xi + \sigma = 4$. We look for the scaling of dimensionless coupling constants, i.e. for dimensionless β -functions. From (47), and using the convention $\eta_k = \partial_t \ln \bar{Z}_k$, one finds:

$$\eta_k = \frac{1}{\bar{Z}_k} \beta(\bar{Z}_k) = \frac{1}{Z_k} \beta(Z_k) + \chi,$$

⁴ Important steps of the calculation are detailed in appendix A

$$\begin{aligned}
\beta(\bar{\mu}_k) &= \frac{1}{\bar{Z}_k l^\chi k^{2-\chi}} \beta(\mu_k) - \eta_k \bar{\mu}_k - (2-\chi) \bar{\mu}_k, \\
\beta(\bar{\lambda}_k) &= \frac{1}{l^\xi k^\sigma \bar{Z}_k^2} \beta(\lambda_k) - 2\eta_k \bar{\lambda}_k - \sigma \bar{\lambda}_k,
\end{aligned} \tag{48}$$

and inserts this in (46) to reach the following expressions:

$$\left\{ \begin{aligned}
\eta_k &= \frac{\bar{\lambda}_k l^\xi k^\sigma}{l^{2\chi} k^{2(2-\chi)} (1 + \bar{\mu}_k)^2} \left\{ (\eta_k - \chi) \left[\frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+1}{2}\right)} \frac{k^{d-1}}{l^{d-1}} + 2(d-1) \frac{k}{l} \right] + 2 \left[(d-1) \frac{k}{l} + \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d-1}{2}\right)} \frac{k^{d-1}}{l^{d-1}} \right] \right\} + \chi \\
\beta(\bar{\mu}_k) &= -\frac{d \bar{\lambda}_k l^\xi k^\sigma}{l^{2\chi} k^{6-2\chi} (1 + \bar{\mu}_k)^2} \left\{ (\eta_k - \chi) \left[\frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+3}{2}\right)} \frac{k^{d+1}}{l^{d-1}} + \frac{4}{3} \frac{k^3}{l} \right] + 2 \left[2 \frac{k^3}{l} + \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+1}{2}\right)} \frac{k^{d+1}}{l^{d-1}} \right] \right\} - \eta_k \bar{\mu}_k - (2-\chi) \bar{\mu}_k \\
\beta(\bar{\lambda}_k) &= \frac{2 \bar{\lambda}_k^2 l^\xi k^\sigma}{l^{2\chi} k^{6-2\chi} (1 + \bar{\mu}_k)^3} \left\{ (\eta_k - \chi) \left[\frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+3}{2}\right)} \frac{k^{d+1}}{l^{d-1}} + \frac{4(2d-1)}{3} \frac{k^3}{l} + 2\delta_{d,3} k^2 \right] \right. \\
&\quad \left. + 2 \left[\frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+1}{2}\right)} \frac{k^{d+1}}{l^{d-1}} + 2(2d-1) \frac{k^3}{l} + 2\delta_{d,3} k^2 \right] \right\} - 2\eta_k \bar{\lambda}_k - \sigma \bar{\lambda}_k
\end{aligned} \right. \tag{49}$$

In order to make the non-compact limit regular, we must solve the system in the variables ξ and χ by requiring that the highest degree of divergence (highest negative power of l) is regularised and all the sub-leading infinities sent to zero. This is achieved by solving, for any $d \geq 3$,

$$\xi - 2\chi - (d-1) = 0. \tag{50}$$

We make a natural choice $\chi = 0$ (thus implying that Z_k is dimensionless), and obtain

$$(\chi = 0, \xi = d-1) \quad \Rightarrow \quad \sigma = 5-d. \tag{51}$$

The resulting system of equations for the theory is:

$$\left\{ \begin{aligned}
\eta_k &= \frac{2\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d-1}{2}\right)} \frac{\bar{\lambda}_k}{(1 + \bar{\mu}_k)^2} \left[\frac{\eta_k}{d-1} + 1 \right] \\
\beta(\bar{\mu}_k) &= \frac{-2d\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+1}{2}\right)} \frac{\bar{\lambda}_k}{(1 + \bar{\mu}_k)^2} \left[\frac{\eta_k}{d+1} + 1 \right] - \eta_k \bar{\mu}_k - 2\bar{\mu}_k \\
\beta(\bar{\lambda}_k) &= \frac{4\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+1}{2}\right)} \frac{\bar{\lambda}_k^2}{(1 + \bar{\mu}_k)^3} \left[\frac{\eta_k}{d+1} + 1 \right] - 2\eta_k \bar{\lambda}_k - (5-d) \bar{\lambda}_k
\end{aligned} \right. \tag{52}$$

which defines an autonomous system of coupled differential equations describing the flow of dimensionless couplings constants.

These equations hold for generic rank d . They could be solved at the same level of generality, in principle, but we find more useful to specialise the analysis for various interesting choices of rank, so that the results can be reported in more explicit terms. Specifically, we study the above system of equations when restricted to the first non-trivial rank situations at $d = 3, 4, 5$. We will analyse the rank $d = 3$ in all details, and, will simply report the key results in higher ranks $d = 4, 5$.

E. Rank $d = 3$

At rank $d = 3$, the system (52) reduces to

$$\begin{cases} \eta_k = \frac{\pi \bar{\lambda}_k}{(1 + \bar{\mu}_k)^2} (\eta_k + 2) \\ \beta(\bar{\mu}_k) = -\frac{3\pi \bar{\lambda}_k}{(1 + \bar{\mu}_k)^2} \left(\frac{\eta_k}{2} + 2 \right) - \eta_k \bar{\mu}_k - 2\bar{\mu}_k \\ \beta(\bar{\lambda}_k) = \frac{\pi \bar{\lambda}_k^2}{(1 + \bar{\mu}_k)^3} (\eta_k + 4) - 2\eta_k \bar{\lambda}_k - 2\bar{\lambda}_k \end{cases} \quad (53)$$

Before proceeding with the standard analysis, which consists in finding fixed points of the flow and studying the linearised equations around them, we point out that, because of the non-linear nature of the β -functions, we have a singularity at $\bar{\mu} = -1$ and $\bar{\lambda} = (1 + \bar{\mu})^2/\pi$. This is a common feature in dealing with a truncated Wetterich equation. In a neighbourhood of those singularities, we do not trust the linear approximation, and being interested in the part of the RG flow connected with the Gaussian fixed point, we will not study the flow beyond the mentioned divergence of the β -functions.

By numerical integration, we find a Gaussian fixed point and three non-Gaussian fixed points in the plane $(\bar{\mu}, \bar{\lambda})$ at:

$$_{d=3}P_1 = (8.619, -47.049), \quad _3P_2 = 10^{-1}(-6.518, 0.096), \quad _3P_3 = 10^{-1}(-8.010, 0.212). \quad (54)$$

A quick inspection proves that $_3P_3$ lies in the sector disconnected from the origin, so we will not perform any analysis around it.

We linearise the system of equations by evaluating the stability matrix around the other three fixed points:

$$\begin{pmatrix} \beta(\bar{\mu}_k) \\ \beta(\bar{\lambda}_k) \end{pmatrix} = \begin{pmatrix} \partial_{\bar{\mu}_k} \beta(\bar{\mu}_k) & \partial_{\bar{\lambda}_k} \beta(\bar{\mu}_k) \\ \partial_{\bar{\mu}_k} \beta(\bar{\lambda}_k) & \partial_{\bar{\lambda}_k} \beta(\bar{\lambda}_k) \end{pmatrix}_{\text{F.P.}} \begin{pmatrix} \bar{\mu}_k \\ \bar{\lambda}_k \end{pmatrix}. \quad (55)$$

In a neighbourhood of the Gaussian fixed point, the stability matrix is of the form

$$(\beta_{ij}^*)|_{GFP} := \begin{pmatrix} -2 & -6\pi \\ 0 & -2 \end{pmatrix} \quad (56)$$

which has an eigenvalue with algebraic multiplicity 2, corresponding to the canonical scaling dimensions of the couplings λ_k and μ_k : $_{d=3}\theta_0 = -2$. The geometric multiplicity of $_3\theta_0$ is 1, hence, the matrix of the linearised system turns out to be not diagonalisable and has a single eigenvector $_3\mathbf{v}_0 = (1, 0)$.

In a neighbourhood of the non-Gaussian fixed points (NGFP) we have:

$$_3P_1 \quad _3\theta_{11} \sim 0.351 \quad \text{for} \quad _3\mathbf{v}_{11} \sim 10^{-1}(0.65, -9.98), \quad (57)$$

$$_3P_1 \quad _3\theta_{12} \sim -2.548 \quad \text{for} \quad _3\mathbf{v}_{12} \sim 10^{-1}(-6.88, 7.26), \quad (58)$$

$$_3P_2 \quad _3\theta_{21} \sim 10.066 \quad \text{for} \quad _3\mathbf{v}_{21} \sim 10^{-1}(9.996, -0.269), \quad (59)$$

$$_3P_2 \quad _3\theta_{22} \sim -1.988 \quad \text{for} \quad _3\mathbf{v}_{22} \sim 10^{-1}(9.987, 0.506). \quad (60)$$

Because of the difference in their magnitudes (distance from the origin), it becomes difficult to plot the two NGFP's simultaneously with enough precision in their vicinity. We plot two sectors of the RG flow in the plane $(\bar{\mu}_k, \bar{\lambda}_k)$ (see Fig.5).

In the vicinity of a fixed point, we define as relevant directions those eigendirections that are UV attractive with respect to the cut-off, while we call irrelevant the UV repulsive eigendirections. Marginal directions can be attractive or repulsive depending on the initial condition of the trajectory. The origin is a *great* attractor and has one relevant direction connecting it to the other two fixed points. The absence of a second eigenvector for the stability matrix around the Gaussian fixed point requires an approximation beyond the linear order, when the flow is studied analytically, and is a signal of the presence of a marginal perturbation. We can instead integrate numerically the flow, and we find that this marginal direction will still be UV attractive, which means that it corresponds to a marginally relevant direction. The fact that the GFP is a sink for the flow, means that this model is asymptotically free with respect to the cut-off. Both non-Gaussian fixed points have one relevant and one irrelevant directions. The eigendirections connecting the three fixed points turn out to be stable under RG transformations and they are characterised by an effect known as

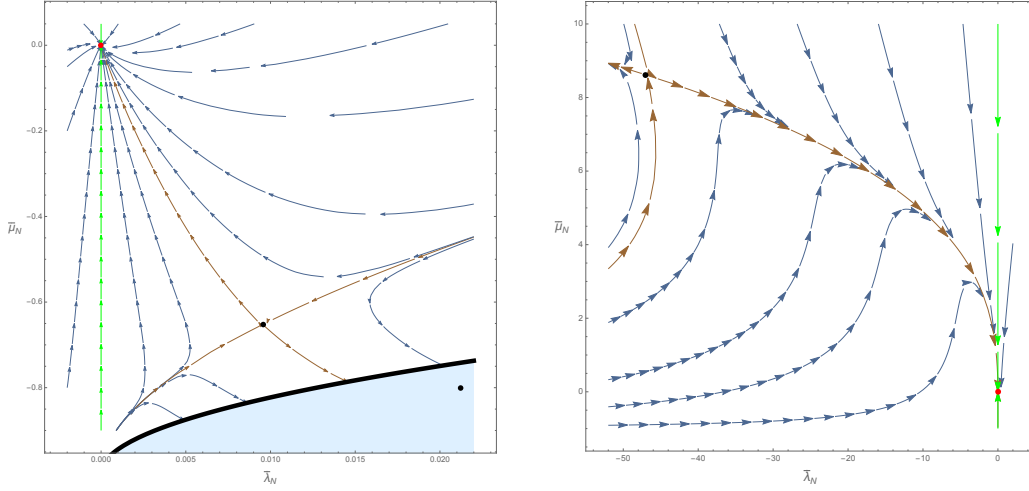


FIG. 5. Flow of the theory. The red and blue lines represent respectively the zeros of $\beta(\bar{\mu}_k)$ and $\beta(\bar{\lambda}_k)$, the brown arrows are the eigenperturbations of the non-Gaussian fixed points (represented in black), and the green ones those of the Gaussian fixed point (in red). Arrows point in the UV direction. The thick black line is the singularity of the flow.

large river effect [52]. This signifies that all the RG trajectories in a neighbourhood of these eigendirections get closer and closer to them while pointing in the UV. This effect shows a splitting of the space of coupling in two regions not connected by any RG trajectory. Thus, the relevant directions for the Gaussian fixed point reflect the properties of a critical surface and suggest the presence of phase transitions in the model. In the $\bar{\lambda}_k > 0$ plane, the flow is similar to the one of standard local scalar field theory in a neighbourhood of the Wilson-Fisher fixed point, but the presence of a second non-Gaussian fixed point in the $\bar{\lambda}_k < 0$ plane makes the theory quite different. Nevertheless, the properties of this second NGFP are basically the same as the former one.

In this context, therefore, we do have strong hint of a phase transition with two phases: a symmetric and a broken one. The spontaneous symmetry breaking would happen while crossing the critical surface, generating a condensed state of the TGFT field (non-zero expectation value of the field operator). This is interesting from a physical perspective, because, in more involved models defined in a simplicial gravity or LQG context, this kind of phase transition has been suggested to relate to the emergence of a geometric spacetime from the theory [12], and the corresponding condensate states have been shown to admit a cosmological interpretation [44]. To confirm this condensate interpretation of the broken phase, one should change parametrisation for the effective potential and study the theory around the new (degenerate) ground state solving the equation of motion in the saddle point approximation. This (complicated) analysis of our TGFT model is left for future work. Here we only notice that, in the constant modes approximation, which forgets about the peculiar combinatorial non-locality of our interactions, and whose results should therefore be taken with great care, we find an algebraic equation of Ginsburg-Landau type for a ϕ^4 scalar complex theory, which indeed describe this type of condensate phase transitions.

F. Rank $d = 4, 5$

We now give a streamlined analysis of the flow in the case of rank $d = 4$, which is very similar to the case $d = 3$, and the rank $d = 5$ which share similarities but also a few differences that we will list.

Writing the system in rank $d = 4$ as

$$\begin{cases} \eta_k = \frac{4\pi}{3} \frac{\bar{\lambda}_k}{(1 + \bar{\mu}_k)^2} (\eta_k + 2) \\ \beta(\bar{\mu}_k) = \frac{-32\pi}{3} \frac{\bar{\lambda}_k}{(1 + \bar{\mu}_k)^2} \left[\frac{\eta_k}{5} + 1 \right] - \eta_k \bar{\mu}_k - 2\bar{\mu}_k \\ \beta(\bar{\lambda}_k) = \frac{16\pi}{3} \frac{\bar{\lambda}_k^2}{(1 + \bar{\mu}_k)^3} \left[\frac{\eta_k}{5} + 1 \right] - 2\eta_k \bar{\lambda}_k - \bar{\lambda}_k \end{cases} \quad (61)$$

we find, in addition to the Gaussian fixed-point, the following NGFPs:

$${}_4P_1 = 10^{-1}(-6.402, 0.058), \quad {}_4P_2 = (1.612, -0.496482), \quad {}_4P_3 = 10^{-1}(-8.452, 0.112). \quad (62)$$

As in the case $d = 3$, the fixed point ${}_4P_3$ lies beyond the singularity. The eigenvalues and eigenvectors in the vicinity of the GFP and of ${}_4P_1$ and ${}_4P_2$ are given in the following table

$$\text{GFP}_4 \quad {}_4\theta_0^+ = -2 \text{ for } {}_4\mathbf{v}_0^+ = (1, 0) \quad (63)$$

$$\text{GFP}_4 \quad {}_4\theta_0^- = -1 \text{ for } {}_4\mathbf{v}_0^- = \left(-\frac{32\pi}{3}, 1\right) \quad (64)$$

$${}_4P_1 \quad {}_4\theta_{11} \sim 7.899 \text{ for } {}_4\mathbf{v}_{11} \sim 10^{-1}(10, -0.106) \quad (65)$$

$${}_4P_1 \quad {}_4\theta_{12} \sim -1.570 \text{ for } {}_4\mathbf{v}_{12} \sim 10^{-1}(10, 0.279), \quad (66)$$

$${}_4P_2 \quad {}_4\theta_{21} \sim -3.082 \text{ for } {}_4\mathbf{v}_{21} \sim 10^{-1}(-10, 0.521) \quad (67)$$

$${}_4P_2 \quad {}_4\theta_{22} \sim 0.439 \text{ for } {}_4\mathbf{v}_{22} \sim 10^{-1}(8.193, -5.733). \quad (68)$$

Negative eigenvalues at the vicinity of the GFP shows that its eigendirections are all relevant. The NGFPs have a relevant and an irrelevant direction.

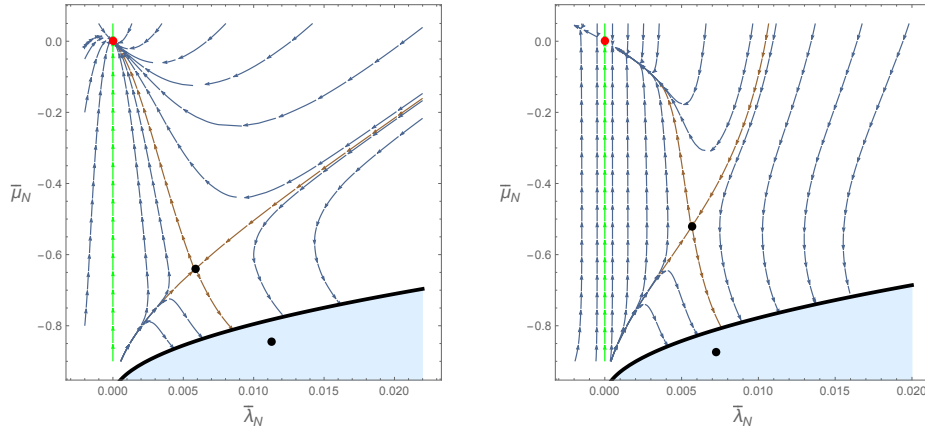


FIG. 6. Flow at rank $d = 4$ (left) and 5 (right).

In rank $d = 5$, on the other hand, the system (52) specialises as

$$\begin{cases} \eta_k = \frac{\pi^2}{2} \frac{\bar{\lambda}_k}{(1 + \bar{\mu}_k)^2} (\eta_k + 2) \\ \beta(\bar{\mu}_k) = -5\pi^2 \frac{\bar{\lambda}_k}{(1 + \bar{\mu}_k)^2} \left[\frac{\eta_k}{6} + 1 \right] - \eta_k \bar{\mu}_k - 2\bar{\mu}_k \\ \beta(\bar{\lambda}_k) = 2\pi^2 \frac{\bar{\lambda}_k^2}{(1 + \bar{\mu}_k)^3} \left[\frac{\eta_k}{6} + 1 \right] - 2\eta_k \bar{\lambda}_k \end{cases} \quad (69)$$

Here, along with the GFP, we identify two NGFPs as

$${}_5P_1 = \left(\frac{-23 + \sqrt{34}}{33}, \frac{4(191 - 4\sqrt{34})}{11979\pi^2} \right) = 10^{-1}(-5.202, 0.056), \quad {}_5P_2 = 10^{-1}(-8.736, 0.072). \quad (70)$$

Again, one of them, ${}_5P_2$, is beyond the singularity so we will skip its analysis. We list eigenvalues and eigenvectors in the vicinity of the GFP and ${}_5P_1$ as follows:

$$\text{GFP}_5 \quad {}_5\theta_0^+ = -2 \text{ for } {}_5\mathbf{v}_0^+ = (1, 0) \quad (71)$$

$$\text{GFP}_5 \quad {}_5\theta_0^- = 0 \text{ for } {}_5\mathbf{v}_0^- = \left(-\frac{5\pi^2}{2}, 1\right) \quad (72)$$

$${}_5P_1 \quad {}_5\theta_1 \sim 2.947 \text{ for } {}_5\mathbf{v}_1 \sim (-249.652, 1) \quad (73)$$

$${}_5P_1 \quad {}_5\theta_2 \sim -0.843 \text{ for } {}_5\mathbf{v}_2 \sim (66.431, 1). \quad (74)$$

The GFP has one relevant eigendirection (corresponding to the mass coupling) and a marginal one. The numerical integration of the flow shows that this direction is marginally relevant for positive λ but become irrelevant for a negative λ . Similar to the previous case, the NGFP ${}_5P_1$ has one relevant and one irrelevant eigendirections.

The models at rank 4 and 5 are very similar to the previous rank 3 case. Hence, similar conclusions concerning the analysis of their flow hold, in particular the separation of the space of couplings in regions which are not connected by any RG trajectories which again suggests of phase transition. The numerical flow of the rank 4 and 5 models have been given in Figure 6. Note that we did not display the NGFP ${}_4P_2$ of the rank 4 model which should be similar to the second fixed point of the rank 3.

IV. GAUGE INVARIANT RANK- d TENSORIAL GROUP FIELD THEORY ON \mathbb{R}

We now proceed to analyse a modified version of the TGFT models studied in the previous section, in which an additional gauge invariance condition is included in the model. These models define topological lattice gauge theories of BF type for the gauge group G at the level of their Feynman amplitudes. The first model of this type has been studied in [40] in rank-6 for the group $G = U(1)$. We therefore extend these first results by working with a non-compact group manifold, albeit still abelian, keeping the rank arbitrary. As for the previous model, we first introduce the gauge-invariant model, then proceed with the FRG analysis in the general case, and finally specialise to interesting choices of rank to show explicitly the results of our analysis.

A. The gauge projection

We work with rank- d fields over the group manifold G satisfying the gauge invariance condition

$$\phi(g_1, g_2, \dots, g_d) = \phi(g_1 h, g_2 h, \dots, g_d h), \quad \forall h \in G. \quad (75)$$

This invariance condition can be imposed directly at the level of the space of fields or as a condition on the dynamics, which then restricts indirectly the field degrees of freedom. In both cases, this translates into a modification of the action (14). This modification can take different forms and should be implemented with some care. A possible (formal) way to implement it would be to allow only the propagation of modes satisfying (75) by inserting in the kinetic kernel a projector on the space of these modes. Defining the projector P and a kinetic kernel \mathcal{K} , one may encounter some ambiguity. A proper inspection shows that in our case, where the kinetic term has the form of a Laplacian plus constant, \mathcal{K} and P commute. We choose to implement the kinetic term of the action in the following form:

$$S[\phi, \bar{\phi}] = \int_{G^{\times d}} [dg_i]_{i=1}^d [dg'_i]_{i=1}^d \bar{\phi}(g_1, g_2, \dots, g_d) (P \cdot \mathcal{K})(\{g_i\}_{i=1}^d; \{g'_i\}_{i=1}^d) \phi(g'_1, g'_2, \dots, g'_d) + \mathcal{V}[\phi, \bar{\phi}], \quad (76)$$

where \mathcal{V} is the interaction term. The main issue of this formulation is that a projector is by definition not invertible, thus, a kinetic kernel built out of such an operator cannot, in general, define a covariance of a field theory measure. We partially avoid this problem by inverting the kinetic kernel in the operatorial sense, in such a way that the same constraint will define the covariance itself. In other words, also the propagator is defined as $P \cdot \mathcal{K}^{-1}$.

Now we restrict our description to the case of the Abelian additive group \mathbb{R} and consider \mathcal{V} with same combinatorics used in section III

$$\begin{aligned} S_1[\phi, \bar{\phi}] &= (2\pi)^{d-1} \int_{\mathbb{R} \times (2d+1)} d\mathbf{x} d\mathbf{y} dh \bar{\phi}(\mathbf{x}) \prod_{i=1}^d \delta(x_i + h - y_i) \left(- \sum_{s=1}^d \Delta_{y_s} + \mu \right) \phi(\mathbf{y}) \\ &\quad + \frac{\lambda}{2} (2\pi)^{2d} \int_{\mathbb{R} \times 2d} d\mathbf{x} d\mathbf{x}' \left[\phi(x_1, x_2, \dots, x_d) \bar{\phi}(x'_1, x'_2, \dots, x'_d) \phi(x'_1, x'_2, \dots, x'_d) \bar{\phi}(x_1, x_2, \dots, x_d) \right. \\ &\quad \left. + \text{sym}\{1, 2, \dots, d\} \right], \end{aligned} \quad (77)$$

where $\mathbf{x} = (x_i)$, $\mathbf{x}' = (x'_i)$ and $\mathbf{y} = (y_i)$ are vectors in \mathbb{R}^d , and $h \in \mathbb{R}$.

We expect that the Wetterich equation will exhibit IR divergences of the same type encountered in the non-projected model, although the gauge invariance conditions relate in a non-trivial way the arguments of the fields entering the interactions, and therefore modifies the combinatorics of the same; as a result, we expect a different degree of IR

divergences with respect to the case we have treated in the previous section. In any case, we introduce again a regularisation scheme. We consider a compact subset D of \mathbb{R} homeomorphic to S^1 and write a regularised action as:

$$\begin{aligned} S_1[\phi, \bar{\phi}] &= (2\pi)^{d-1} \int_{D \times (2d+1)} d\mathbf{x} d\mathbf{y} dh \bar{\phi}(\mathbf{x}) \prod_{i=1}^d \delta(x_i + h - y_i) \left(- \sum_{s=1}^d \Delta_{y_s} + \mu \right) \phi(\mathbf{y}) \\ &+ \frac{\lambda}{2} (2\pi)^{2d} \int_{D \times 2d} d\mathbf{x} d\mathbf{x}' \left[\phi(x_1, x_2, \dots, x_d) \bar{\phi}(x'_1, x'_2, \dots, x'_d) \phi(x'_1, x'_2, \dots, x'_d) \bar{\phi}(x_1, x_2, \dots, x_d) \right. \\ &\left. + \text{sym}\{1, 2, \dots, d\} \right], \end{aligned} \quad (78)$$

where we used the same notations introduced in section III C.

The computation will be performed in momentum space. Using again the same notation for the lattice as $D^* = \mathcal{D}^{\times d}$, and denoting the gauge invariance constraint on the corresponding lattice as $\delta_{\mathcal{D}}(X) := \delta_{\mathcal{D}}(X, 0)$, the Fourier series of the model (78) reads:

$$S_1[\phi, \bar{\phi}] = l^d \sum_{\mathbf{p} \in D^*} \bar{\phi}(\mathbf{p}) \left[\Sigma_s p_s^2 + \mu \right] \phi(\mathbf{p}) \delta_{\mathcal{D}}(\Sigma p) + \frac{\lambda}{2} l^{2d} \sum_{\mathbf{p}, \mathbf{p}' \in D^*} \left[\phi_{12\dots d} \bar{\phi}_{1'2'\dots d'} \phi_{1'2'\dots d'} \bar{\phi}_{12\dots d} + \text{sym}\{1, 2, \dots, d\} \right]. \quad (79)$$

The general FRG formalism introduced in section II B applies to this model as to the one in the previous section. In particular, the regulator kernel will incorporate the same gauge constraint appearing in the kinetic term. The Wetterich equation has the same structure as well and expands again as (30).

We choose to truncate the effective action as:

$$\begin{aligned} \Gamma_k^1[\varphi, \bar{\varphi}] &= l^d \sum_{\mathbf{p} \in D^*} \bar{\varphi}(\mathbf{p}) \left[Z_k \Sigma_s p_s^2 + \mu_k \right] \varphi(\mathbf{p}) \delta_{\mathcal{D}}(\Sigma p) \\ &+ \frac{\lambda_k}{2} l^{2d} \sum_{\mathbf{p}, \mathbf{p}' \in D^*} \left[\varphi_{12\dots d} \bar{\varphi}_{1'2'\dots d'} \varphi_{1'2'\dots d'} \bar{\varphi}_{12\dots d} + \text{sym}\{1, 2, \dots, d\} \right], \end{aligned} \quad (80)$$

and, then, we introduce the kernels (using the same notation as (30)):

$$R_k(\mathbf{q}, \mathbf{q}') = \Theta(k^2 - \Sigma_s q_s^2) Z_k (k^2 - \Sigma_s q_s^2) \delta_{\mathcal{D}}(\Sigma q) \prod \delta_{\mathcal{D}}(\mathbf{q}, \mathbf{q}'), \quad (81)$$

$$F_k^1(\mathbf{q}, \mathbf{q}') = \frac{\delta^2}{\delta \bar{\varphi}_{\mathbf{q}'} \delta \varphi_{\mathbf{q}}} \mathcal{V}^1[\varphi, \bar{\varphi}], \quad (82)$$

where \mathcal{V}_k^1 refers to the interaction part of Γ_k^1 . This is a natural choice following directly from a straightforward FRG formulation of (78). Performing the computation of the Wetterich equation, however, one realises that this proposal drastically fails: the delta's enforcing the gauge constraints do not convolute properly with the TGFT fields. This is due to the fact that, if one evaluates (30) using (81) and (82), the fields appearing in the r.h.s. come from the F_k^1 operator, while the constraints always come from the mass-like terms. The comparison of the two sides of the Wetterich equation for this model, then would lead to all β -functions being trivial.

A moment of reflection shows that another way of choosing the interaction term produces a more sensible result. We simply insert gauge projections also in all fields in the interaction. An interaction satisfying this requirement expresses as:

$$\begin{aligned} \mathcal{V}[\phi, \bar{\phi}] &= \frac{\lambda_k}{2} (2\pi)^{2d-4} \int_{D \times (6d+4)} \{d\mathbf{w}^i\}_{i=1}^4 d\mathbf{x} d\mathbf{x}' \{dh_j\}_{j=1}^4 \phi(\mathbf{w}^1) \bar{\phi}(\mathbf{w}^2) \phi(\mathbf{w}^3) \bar{\phi}(\mathbf{w}^4) \\ &\times \delta(x_1 + h_1 - w_1^1) \delta(x_2 + h_1 - w_2^1) \dots \delta(x_d + h_1 - w_d^1) \\ &\times \delta(x'_1 + h_2 - w_1^2) \delta(x_2 + h_2 - w_2^2) \dots \delta(x_d + h_2 - w_d^2) \\ &\times \delta(x'_1 + h_3 - w_1^3) \delta(x'_2 + h_3 - w_2^3) \dots \delta(x'_d + h_3 - w_d^3) \\ &\times \delta(x_1 + h_4 - w_1^4) \delta(x'_2 + h_4 - w_2^4) \dots \delta(x'_d + h_4 - w_d^4) \\ &+ \text{sym}\{1, 2, \dots, d\} \\ &= \frac{\lambda_k}{2} l^{2d} \sum_{\mathbf{p}, \mathbf{p}'} \phi_{12\dots d} \bar{\phi}_{1'2'\dots d'} \phi_{1'2'\dots d'} \bar{\phi}_{12\dots d} \delta_{\mathcal{D}}(\Sigma p) \delta_{\mathcal{D}}(\Sigma p') \\ &\times \delta_{\mathcal{D}}(p'_1 + p_2 + \dots + p_d) \delta_{\mathcal{D}}(p_1 + p'_2 + \dots + p'_d) + \text{sym}\{1, 2, \dots, d\}. \end{aligned} \quad (83)$$

Hence, re-starting the analysis from the beginning, we define a model with gauge constraints on both the kinetic and interaction kernels via the action:

$$\begin{aligned}
S[\phi, \bar{\phi}] &= l^{2d} \sum_{\mathbf{p}} \bar{\phi}(\mathbf{p}) \left[\Sigma_s p_s^2 + \mu \right] \phi(\mathbf{p}) \delta_{\mathcal{D}}(\Sigma p) \\
&+ \frac{\lambda_k}{2} l^{2d} \sum_{\mathbf{p}, \mathbf{p}'} \phi_{12\dots d} \bar{\phi}_{1'2'\dots d} \phi_{1'2'\dots d'} \bar{\phi}_{12'\dots d'} \delta_{\mathcal{D}}(\Sigma p) \delta_{\mathcal{D}}(\Sigma p') \delta_{\mathcal{D}}(p'_1 + p_2 + \dots + p_d) \delta_{\mathcal{D}}(p_1 + p'_2 + \dots + p'_d) \\
&+ \text{sym} \left\{ 1, 2, \dots, d \right\},
\end{aligned} \tag{84}$$

with corresponding continuous model defined by

$$\begin{aligned}
S[\phi, \bar{\phi}] &= \int d\mathbf{p} \bar{\phi}(\mathbf{p}) \left[\Sigma_s p_s^2 + \mu \right] \phi(\mathbf{p}) \delta(\Sigma p) \\
&+ \frac{\lambda_k}{2} \int d\mathbf{p} d\mathbf{p}' \phi_{12\dots d} \bar{\phi}_{1'2'\dots d} \phi_{1'2'\dots d'} \bar{\phi}_{12'\dots d'} \delta_{\mathcal{D}}(\Sigma p) \delta_{\mathcal{D}}(\Sigma p') \delta_{\mathcal{D}}(p'_1 + p_2 + \dots + p_d) \delta_{\mathcal{D}}(p_1 + p'_2 + \dots + p'_d) \\
&+ \text{sym} \left\{ 1, 2, \dots, d \right\}.
\end{aligned} \tag{85}$$

In fact, with hindsight, one realises that this result could have been guessed from a more general consideration. Even if the perturbative quantum amplitudes of the theory do not depend on whether the gauge projection appears in the kinetic term, in the interaction or in both, and only gauge invariant degrees of freedom have non trivial Feynman amplitudes (spin foam models) the non-perturbative analysis is of course radically different. From a non-perturbative point of view one is suggested to simply project the model to the space of gauge invariant fields, and thus insert projections in all elements of the TGFT action. From this point of view, a model which presents this constraint in only one of the two terms cannot be consistent. This directly reflects in the analysis we just presented.

At the same time, notice that inserting gauge projections on all fields in the action, both in kinetic and interaction terms, results in a trivial overall divergence equal to the volume of the domain, due to the fact that the combinatorics of field pairings is such that imposing gauge invariance on all but one field in each polynomial automatically implies the gauge invariance of the last one. We can easily remove this trivial divergence, therefore, by removing one gauge projection from one of the fields in each polynomial term in the action. The above prescription of the effective action together with (29), coincides with the Wetterich equation as formulated in [40] (albeit the formalism differs by the nature of the field background).

We can now proceed further using the model (85).

B. Effective action and Wetterich equation

Having defined the main ingredients of the model, we are in position to analyse its FRG equation. We shall again restrict to a simple truncation of the effective action for the model (84), which reads:

$$\begin{aligned}
\Gamma_k[\varphi, \bar{\varphi}] &= l^d \sum_{\mathbf{p}} \bar{\varphi}(\mathbf{p}) \left[Z_k \Sigma_s p_s^2 + \mu_k \right] \varphi(\mathbf{p}) \delta_{\mathcal{D}}(\Sigma p) \\
&+ \frac{\lambda_k}{2} l^{2d} \sum_{\mathbf{p}, \mathbf{p}'} \varphi_{12\dots d} \bar{\varphi}_{1'2'\dots d} \varphi_{1'2'\dots d'} \bar{\varphi}_{12'\dots d'} \delta_{\mathcal{D}}(\Sigma p) \delta_{\mathcal{D}}(\Sigma p') \\
&\times \delta_{\mathcal{D}}(p'_1 + p_2 + \dots + p_d) \delta_{\mathcal{D}}(p_1 + p'_2 + \dots + p'_d) + \text{sym} \left\{ 1, 2, \dots, d \right\}.
\end{aligned} \tag{86}$$

Considering that $[\delta_{\mathcal{D}}(p)] = -1$, the dimensional analysis for the coupling constants gives different results from the model of section III. We have:

$$\begin{aligned}
[Z_k] = 0 &\Rightarrow [\mu_k] = 2 \\
2[\varphi] + d + 2 - 1 = 0 &\Rightarrow [\varphi] = -\frac{d+1}{2} \\
[\lambda_k] + 2d + 4[\varphi] - 4 = 0 &\Rightarrow [\lambda_k] = 6,
\end{aligned} \tag{87}$$

where, again, we set the canonical dimensions by requiring $[S] = [\Gamma_k] = 0$ and $[dp] = 1$.

We introduce:

$$R_k(\mathbf{q}, \mathbf{q}') = \Theta(k^2 - \Sigma_s q_s^2) Z_k(k^2 - \Sigma_s q_s^2) \delta_{\mathcal{D}}(\Sigma q) \delta_{\mathcal{D}^*}(\mathbf{q}, \mathbf{q}'), \tag{88}$$

$$\partial_t R_k(\mathbf{q}, \mathbf{q}') = \Theta(k^2 - \Sigma_s q_s^2) [\partial_t Z_k(k^2 - \Sigma_s q_s^2) + 2k^2 Z_k] \delta_{\mathcal{D}}(\Sigma q) \delta_{D^*}(\mathbf{q}, \mathbf{q}'), \quad (89)$$

$$\begin{aligned} F_k(\mathbf{q}, \mathbf{q}') &= \lambda_k \left[l^{d-1} \sum_{m_i} \varphi_{q'_1 m_2 \dots m_d} \bar{\varphi}_{q_1 m_2 \dots m_d} \delta_{\mathcal{D}}(\Sigma q) \delta_{\mathcal{D}}(q'_1 + m_2 + \dots + m_d) \right. \\ &\quad \times \delta_{\mathcal{D}}(q'_1 + q_2 + \dots + q_d) \delta_{\mathcal{D}}(q_1 + m_2 + \dots + m_d) \delta_{\mathcal{D}}(q_2 - q'_2) \dots \delta_{\mathcal{D}}(q_d - q'_d) \\ &\quad + l \sum_{m_1} \varphi_{m_1 q'_2 \dots q'_d} \bar{\varphi}_{m_1 q_2 \dots q_d} \delta_{\mathcal{D}}(\Sigma q) \delta_{\mathcal{D}}(m_1 + q'_2 + \dots + q'_d) \\ &\quad \times \delta_{\mathcal{D}}(m_1 + q_2 + \dots + q_d) \delta_{\mathcal{D}}(q_1 + q'_2 + \dots + q'_d) \delta_{\mathcal{D}}(q_1 - q'_1) \\ &\quad \left. + \text{sym}\{1, 2, \dots, d\} \right], \quad (90) \end{aligned}$$

$$P_k(\mathbf{q}, \mathbf{q}') = R_k(\mathbf{q}, \mathbf{q}') + \left(Z_k \sum_s q_s^2 + \mu_k \right) \delta_{\mathcal{D}}(\Sigma q) \delta_{D^*}(\mathbf{q}, \mathbf{q}'). \quad (91)$$

This leads to the Wetterich equation:

$$\begin{aligned} \partial_t \Gamma_k &= \text{Tr}[\partial_t R_k \cdot (P_k + F_k)^{-1}] \\ &= l^{2d} \sum_{\mathbf{p}, \mathbf{p}'} \partial_t R_k(\mathbf{p}, \mathbf{p}') (P_k + F_k)^{-1}(\mathbf{p}', \mathbf{p}). \quad (92) \end{aligned}$$

On the left hand side, as in section III, we truncate at the level of the quartic interactions. This gives then, for the r.h.s. of the Wetterich equation, the same expansion shown in (30), where now the operators involved are given by (89), (90) and (91).

An extra subtlety must be paid attention to, however, in extracting the β -functions of this model. The δ 's implementing the convolutions which appear in the P_k operators can be inverted using (37), and summing over their indices we do not modify the dimensions of the whole expression. This is, however, not true for the δ 's coming from the gauge constraints because they are not summed, so we need to keep them in the denominator. In any case, these constraints, turn out to be redundant with other delta functions coming from the F_k and $\partial_t R_k$ operators, in such a way that their contribution, because of the regularization, is equivalent to some power of l , and it is naturally well defined.

C. β -functions and RG flows

Expanding the FRG equation (92), we find the following system of dimensionful β -functions (the main steps of the calculations are given in appendix B):

$$\left\{ \begin{aligned} \beta_{d \neq 4}(Z_k) &= \frac{d\lambda_k}{(Z_k k^2 + \mu_k)^2} \left\{ \partial_t Z_k \left[\frac{\pi^{\frac{d-2}{2}}}{(d-1)^{\frac{3}{2}} \Gamma_E\left(\frac{d}{2}\right)} \frac{k^{d-2}}{l^d} + \frac{1}{l^2} \right] + \frac{2\pi^{\frac{d-2}{2}} Z_k}{(d-1)^{\frac{3}{2}} \Gamma_E\left(\frac{d-2}{2}\right)} \frac{k^{d-2}}{l^d} \right\} \\ \beta_{d \neq 4}(\mu_k) &= -\frac{d\lambda_k}{(Z_k k^2 + \mu_k)^2} \left\{ \partial_t Z_k \left[\frac{k^d}{l^d} \frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1} \Gamma_E\left(\frac{d+2}{2}\right)} + \frac{k^2}{l^2} \right] + 2Z_k \left[\frac{k^d}{l^d} \frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1} \Gamma_E\left(\frac{d}{2}\right)} + \frac{k^2}{l^2} \right] \right\} \\ \beta_{d \neq 4}(\lambda_k) &= \frac{2\lambda_k^2}{(Z_k k^2 + \mu_k)^3} \left\{ \partial_t Z_k \left[\frac{2\pi^{\frac{d-2}{2}}}{d\sqrt{d-1} \Gamma\left(\frac{d}{2}\right)} \frac{k^d}{l^d} + (2[d + \delta_{d,3}] - 1) \frac{k^2}{l^2} \right] + 2Z_k \left[\frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1} \Gamma\left(\frac{d}{2}\right)} \frac{k^d}{l^d} + (2[d + \delta_{d,3}] - 1) \frac{k^2}{l^2} \right] \right\} \end{aligned} \right. \quad (93)$$

and, at $d = 4$, we have

$$\left\{ \begin{aligned} \beta_{d=4}(Z_k) &= \frac{\lambda_k}{(Z_k k^2 + \mu_k)^2} \left\{ \partial_t Z_k \left[\frac{\pi}{\sqrt{3}} \frac{k^2}{l^4} + \frac{4}{l^2} \right] + \frac{2\pi}{\sqrt{3}} \frac{k^2}{l^4} Z_k \right\} \\ \beta_{d=4}(\mu_k) &= -\frac{4\lambda_k}{(Z_k k^2 + \mu_k)^2} \left\{ \partial_t Z_k \left[\frac{\pi}{2\sqrt{3}} \frac{k^4}{l^4} + \frac{k^2}{l^2} \right] + 2Z_k \left[\frac{\pi}{\sqrt{3}} \frac{k^4}{l^4} + \frac{k^2}{l^2} \right] \right\} \\ \beta_{d=4}(\lambda_k) &= \frac{2\lambda_k^2}{(Z_k k^2 + \mu_k)^3} \left\{ \partial_t Z_k \left[\frac{2\pi}{4\sqrt{3}} \frac{k^4}{l^4} + 7 \frac{k^2}{l^2} \right] + 2Z_k \left[\frac{\pi}{\sqrt{3}} \frac{k^4}{l^4} + 7 \frac{k^2}{l^2} \right] \right\} \end{aligned} \right. \quad (94)$$

In order to obtain a well defined non-compact limit of the model, we use a modified ansatz (different from the one of section III D):

$$Z_k = \bar{Z}_k k^{-\chi} l^\chi, \quad \mu_k = \bar{\mu}_k \bar{Z}_k k^{2-\chi} l^\chi, \quad \lambda_k = \bar{\lambda}_k \bar{Z}_k^2 k^{6-\xi} l^\xi, \quad (95)$$

from which we obtain the dimensionless β -functions according to the following calculation:

$$\begin{aligned} \eta_k &= \frac{1}{\bar{Z}_k} \beta(\bar{Z}_k) = \frac{k^\chi l^{-\chi}}{\bar{Z}_k} \beta(Z_k) + \chi, \\ \beta(\bar{\mu}_k) &= \frac{k^{\chi-2} l^{-\chi}}{\bar{Z}_k} \beta(\mu_k) - \eta_k \bar{\mu}_k + (\chi - 2) \bar{\mu}_k, \\ \beta(\bar{\lambda}_k) &= \frac{k^{\xi-6} l^{-\xi}}{\bar{Z}_k^2} \beta(\lambda_k) - 2\eta_k \bar{\lambda}_k + (\xi - 6) \bar{\lambda}_k. \end{aligned} \quad (96)$$

Inserting the above in (93), we deduce the equations for the dimensionless coupling constants:

$$\begin{aligned} \eta_k &= \frac{d \bar{\lambda}_k k^{2-\xi+2\chi} l^{\xi-2\chi}}{(1 + \bar{\mu}_k)^2} \left\{ (\eta_k - \chi) \left[\frac{\pi^{\frac{d-2}{2}}}{(d-1)^{\frac{3}{2}} \Gamma_E\left(\frac{d}{2}\right)} \frac{k^{d-2}}{l^d} + \frac{1}{l^2} \right] + \frac{2 \pi^{\frac{d-2}{2}}}{(d-1)^{\frac{3}{2}} \Gamma_E\left(\frac{d-2}{2}\right)} \frac{k^{d-2}}{l^d} \right\} + \chi \\ \beta_{d \neq 4}(\bar{\mu}_k) &= -\frac{d \bar{\lambda}_k k^{-\xi+2\chi} l^{\xi-2\chi}}{(1 + \bar{\mu}_k)^2} \left\{ (\eta_k - \chi) \left[\frac{k^d}{l^d} \frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1} \Gamma_E\left(\frac{d+2}{2}\right)} + \frac{k^2}{l^2} \right] + 2 \left[\frac{k^d}{l^d} \frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1} \Gamma_E\left(\frac{d}{2}\right)} + \frac{k^2}{l^2} \right] \right\} \\ &\quad - \eta_k \bar{\mu}_k + (\chi - 2) \bar{\mu}_k \\ \beta_{d \neq 4}(\bar{\lambda}_k) &= \frac{2 \bar{\lambda}_k^2 k^{2\chi-\xi} l^{\xi-2\chi}}{(1 + \bar{\mu}_k)^3} \left\{ (\eta_k - \chi) \left[\frac{2 \pi^{\frac{d-2}{2}}}{d \sqrt{d-1} \Gamma\left(\frac{d}{2}\right)} \frac{k^d}{l^d} + (2[d + \delta_{d,3}] - 1) \frac{k^2}{l^2} \right] \right. \\ &\quad \left. + 2 \left[\frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1} \Gamma\left(\frac{d}{2}\right)} \frac{k^d}{l^d} + (2[d + \delta_{d,3}] - 1) \frac{k^2}{l^2} \right] \right\} - 2\eta_k \bar{\lambda}_k + (\xi - 6) \bar{\lambda}_k \end{aligned} \quad (97)$$

As in section III D, the system of β -functions is non-autonomous in the IR cut-off k , as long as l is kept finite. We also notice a different dependence on the parameters k and l with respect to (49). The difference is of course a consequence of the presence of the delta functions which, having non-trivial dimensions, change both the canonical and scaling dimensions of couplings and fields, and remove degrees of freedom from the space of dynamical fields by imposing the gauge invariance constraints. Concerning this, we point out that, had we introduced one delta for each field appearing in both the kinetic and interaction kernels, this operation would have caused some extra divergences, but it would have also allowed us to absorb, from the point of view of the dimensions, the contribution of deltas inside a redefinition of the fields. In that case we would expect the couplings to have the same (canonical) dimensions of those appearing in the previous model. Finally, we can also note that the system might be re-expressed in terms of a shifted anomalous dimension $\eta_k \rightarrow \eta_k - \chi$, thus it could be defined up to constant χ . In the following, we have set $\chi = 0$.

To get an autonomous system in the limit of the regulator being removed, we set

$$\xi - 2\chi - d = 0, \quad (98)$$

and fixing $\chi = 0$, we come to $\xi = d$. In the thermodynamic limit, for $d \neq 4$, we obtain the autonomous system,

$$\begin{cases} \eta_k = \frac{d \bar{\lambda}_k}{(1 + \bar{\mu}_k)^2} \frac{\pi^{\frac{d-2}{2}}}{(d-1)^{\frac{3}{2}}} \left\{ \eta_k \frac{1}{\Gamma_E\left(\frac{d}{2}\right)} + \frac{2}{\Gamma_E\left(\frac{d-2}{2}\right)} \right\} \\ \beta_{d \neq 4}(\bar{\mu}_k) = -\frac{d \bar{\lambda}_k}{(1 + \bar{\mu}_k)^2} \frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1}} \left\{ \eta_k \frac{1}{\Gamma_E\left(\frac{d+2}{2}\right)} + \frac{2}{\Gamma_E\left(\frac{d}{2}\right)} \right\} - (\eta_k + 2) \bar{\mu}_k \\ \beta_{d \neq 4}(\bar{\lambda}_k) = \frac{2 \bar{\lambda}_k^2}{(1 + \bar{\mu}_k)^3} \frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1}} \left\{ \eta_k \frac{1}{\Gamma_E\left(\frac{d+2}{2}\right)} + \frac{2}{\Gamma_E\left(\frac{d}{2}\right)} \right\} - 2\eta_k \bar{\lambda}_k + (d - 6) \bar{\lambda}_k \end{cases} \quad (99)$$

In passing, we observe that at $d = 6 = \xi$, the coupling λ_k becomes marginal.

The same analysis performed at $d = 4$ yields

$$\begin{cases} \eta_k = \frac{\bar{\lambda}_k}{(1 + \bar{\mu}_k)^2} \frac{\pi}{\sqrt{3}} (\eta_k + 2) \\ \beta_{d=4}(\mu_k) = -\frac{4\bar{\lambda}_k}{(1 + \bar{\mu}_k)^2} \frac{\pi}{\sqrt{3}} \left(\frac{1}{2}\eta_k + 2\right) - (\eta_k + 2)\bar{\mu}_k \\ \beta_{d=4}(\lambda_k) = \frac{2\bar{\lambda}_k^2}{(1 + \mu_k)^3} \frac{\pi}{\sqrt{3}} \left(\frac{1}{2}\eta_k + 2\right) - 2(\eta_k + 1)\bar{\lambda}_k \end{cases} \quad (100)$$

D. Rank $d = 3, 4$

We can now fix the rank d , to be able to explicitly compute the flow.

We start with the case $d = 3$. The dependence in χ can be re-absorbed by a redefinition $\eta_k \rightarrow \eta_k - \chi$ (and the resulting variable is called again η_k). We therefore have finally a system of dimensionless β -functions given by

$$\begin{cases} \eta_k = \frac{3\bar{\lambda}_k}{\sqrt{2}(1 + \bar{\mu}_k)^2 - 3\bar{\lambda}_k} \\ \beta(\bar{\mu}_k) = -\frac{6\bar{\lambda}_k\sqrt{2}}{(1 + \bar{\mu}_k)^2} \left(\frac{\eta_k}{3} + 1\right) - \eta_k\bar{\mu}_k - 2\bar{\mu}_k \\ \beta(\bar{\lambda}_k) = \frac{4\bar{\lambda}_k^2\sqrt{2}}{(1 + \bar{\mu}_k)^3} \left(\frac{\eta_k}{3} + 1\right) - 2\eta_k\bar{\lambda}_k - 3\bar{\lambda}_k \end{cases} \quad (101)$$

Like in the model without gauge projection, the system presents a divergence in the flow due to the truncation scheme. Here the singularity occurs at $\bar{\mu} = -1$ and $\bar{\lambda} = \frac{\sqrt{2}}{3}(1 + \bar{\mu})^2$. In the plane $(\bar{\mu}, \bar{\lambda})$, we find four fixed points, the Gaussian (GFP) and three non-Gaussian fixed points (NGFP) at:

$${}_3P_1 = (10)^{-1}(-7.083, 0.154), \quad {}_3P_2 = 10^{-1}(-7.935, 0.273), \quad {}_3P_3 = (-12.809, 169.635). \quad (102)$$

Both ${}_3P_2$ and ${}_3P_3$ lie in the sector disconnected from the origin, therefore we restrict the analysis and linearize the system only around ${}_3P_1$ and the Gaussian fixed point. The following eigenvalues and eigenvectors can be found by calculation from the stability matrix:

$$\text{GFP}_3 \quad {}_3\theta_0^+ = -2 \quad \text{for} \quad {}_3\mathbf{v}_0^+ = (1, 0), \quad (103)$$

$$\text{GFP}_3 \quad {}_3\theta_0^- = -3 \quad \text{for} \quad {}_3\mathbf{v}_0^- = (6\sqrt{2}, 1), \quad (104)$$

$${}_3P_1 \quad {}_3\theta_1 \sim 14.47 \quad \text{for} \quad {}_3\mathbf{v}_1 \sim 10^{-1}(9.986, -0.529), \quad (105)$$

$${}_3P_1 \quad {}_3\theta_2 \sim -2.29 \quad \text{for} \quad {}_3\mathbf{v}_2 \sim 10^{-1}(9.948, 1.022). \quad (106)$$

Negative eigenvalues represents UV-attractive eigendirections, while positive eigenvalues correspond to UV-repulsive eigendirections. From the plot in Fig.7, we see that the Gaussian fixed point, where we have two negative eigenvalues corresponding to the scaling dimensions of the couplings, is a UV-attractor and has two relevant directions. Thus, we infer that the model is asymptotically free in the UV. Meanwhile, the NGFP has one relevant direction and one irrelevant direction. In this model, there are no marginal directions in the flow and, qualitatively, the structure of the plot is again reminiscent of the Wilson-Fisher fixed point in standard scalar field theory in three dimensions. This is again a strong hint to a phase transition between a symmetric and a broken phase, interpreted as a condensate phase labeled by a non-zero expectation value of the TGFT field operator.

Comparing this model with the one studied in section III E, we can list some similarities, as well as the differences that follow then directly from the new gauge invariance imposition.

From the computational point of view, there are no fundamental differences. The presence of the gauge constraints influences the end result for what concerns the exact dependence of the FRG equations on the parameters k and l . The way the thermodynamic limit turns the regularized system of RG equations into an autonomous one is similar, but resulting from different canonical dimensions attributed to the various elements of the theory. For example, the canonical dimension of the ϕ^4 -coupling changes from one model to the other. We claim that these models are not in the same universality class.

From a qualitative point of view, we find in both models the same number of non-Gaussian fixed points, but their distribution in the plane $(\bar{\mu}, \bar{\lambda})$ is different. The TGFT model without gauge projection has two interesting NGFPs in

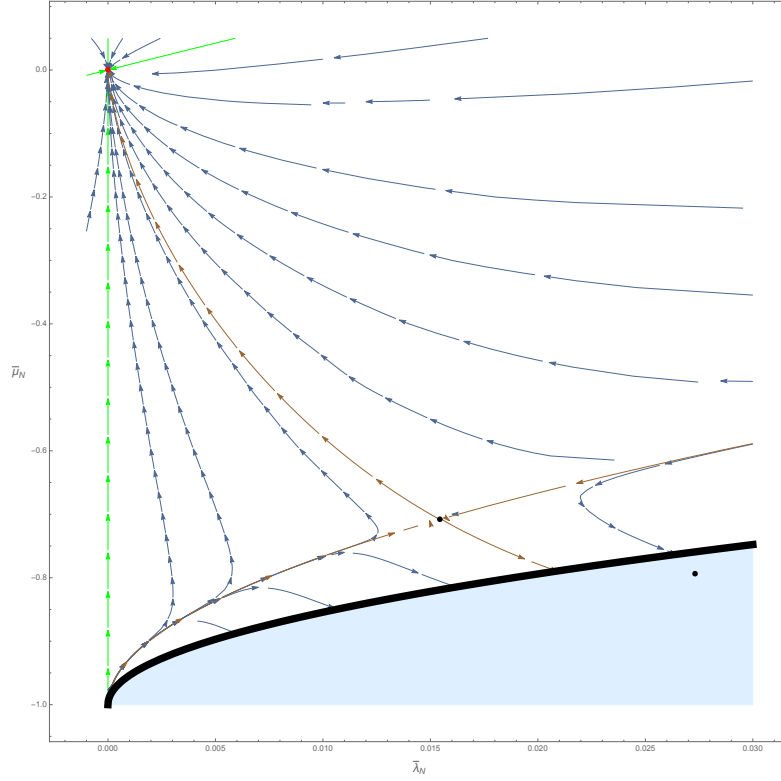


FIG. 7. Flow for the rank-3 gauge invariant model. Brown arrows represent the eigendirections of the NGFP (in black), while green arrows are the eigendirections of GFP (in red). The thick black line indicates the singularity of the system.

the region of the plane $(\bar{\mu}, \bar{\lambda})$ connected to the origin, whereas the gauge projected model has a unique NGFP lying in the same region. Also, the linearised theory around the Gaussian fixed point turns out to be slightly different. While in the previous section we have found a non-diagonalizable stability matrix with only one strictly relevant direction, for the gauge invariant model we have two relevant directions and the eigenperturbations form indeed a basis for the linearised system. On the other hand, the GFPs of both models are sinks, and so both models are asymptotically free.

In rank $d = 4$, the results are very similar to the above rank $d = 3$. We obtain, in addition to the Gaussian fixed point, the fixed points

$${}_4P_1 = (10)^{-1}(-7.05, 0.093), \quad {}_4P_2 = 10^{-1}(-8.465, 0.228), \quad {}_4P_3 = (10.051, -97.962). \quad (107)$$

${}_4P_2$ which stands below the singularity will be not further analysed. We will focus on the rest of the fixed points and perform a linearisation around those.

Around the Gaussian fixed point the stability matrix becomes

$$(\beta_{ij}^*)|_{GFP} := \begin{pmatrix} -2 & -\frac{8\pi}{\sqrt{3}} \\ 0 & -2 \end{pmatrix} \quad (108)$$

which has an eigenvalue ${}_4\theta_0 = -2$ with multiplicity 2 with a single eigenvector ${}_4\mathbf{v}_0 = (1, 0)$. We cannot diagonalise it and will integrate numerically the flow around this point.

We have the following critical exponents:

$$\text{GFP}_4 \quad {}_4\theta_0 = -2 \text{ for } {}_4\mathbf{v}_0 = (1, 0), \quad (109)$$

$${}_4P_1 \quad {}_4\theta_{11} \sim 11.819 \text{ for } {}_4\mathbf{v}_{11} \sim 10^{-1}(10, -0.225), \quad (110)$$

$${}_4P_1 \quad {}_4\theta_{12} \sim -2.158 \text{ for } {}_4\mathbf{v}_{12} \sim 10^{-1}(10, 0.624), \quad (111)$$

$${}_4P_3 \quad {}_4\theta_{31} \sim -2.654 \text{ for } {}_4\mathbf{v}_{31} \sim 10^{-1}(-3.891, 9.211), \quad (112)$$

$${}_4P_3 \quad {}_4\theta_{32} \sim 0.624 \text{ for } {}_4\mathbf{v}_{32} \sim 10^{-1}(0.316, -10). \quad (113)$$

Both NGFPs have one relevant and one irrelevant directions. The analysis of perturbations around the fixed points leads to the phase diagram and RG flow presented in Fig.6. From the numerical integration, we observe that the second eigendirection of the GFP is marginally relevant. We represent the phase diagram in Fig.6.

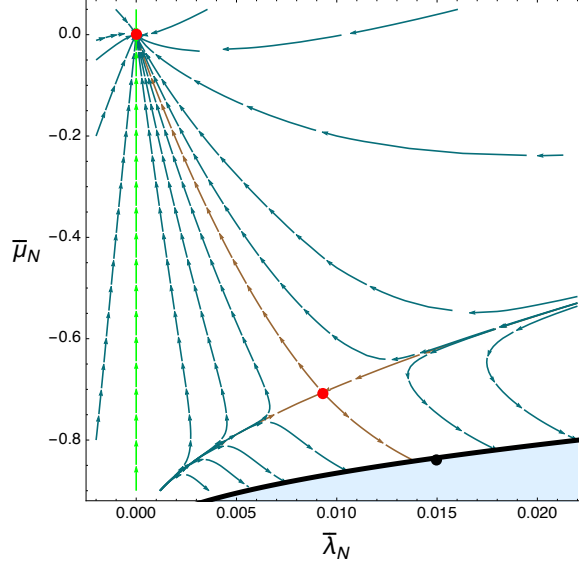


FIG. 8. Flow of the gauged model at rank $d = 4$.

We see once more RG trajectories indicating asymptotic freedom in the UV, and the presence of a phase transition between a symmetric and a broken phase in the IR.

E. Rank $d = 6$

Another interesting case to look at in more detail is the one for $d = 6$. For this rank, the model has one marginal direction around the GFP as the scaling dimension of the coupling λ vanishes. In this case, in fact, we can compare our results directly with the ones obtained in [40]. This comparison has two aspects. At the regularised level, with the system restricted to (six copies of) the compact domain S^1 , we expect our RG equations to match the ones found in [40], up to normalisations. This can indeed be verified, but we do not report on it. On the other hand, by studying the RG flow in the thermodynamic limit, we will then be able to check how the phase diagram we obtain compares with the limiting cases studied for the compact model, expecting a qualitative agreement with the results found there in the UV approximation.

In rank $d = 6$, we have the following fixed points alongside the Gaussian fixed point:

$${}_6P_{\pm} = \left(\frac{1}{234}(-175 \pm \sqrt{1141}), \frac{\sqrt{5}(43309 \mp 79\sqrt{1141})}{1067742\pi^2} \right) \quad (114)$$

The NGFP ${}_6P_-$ is below the singularity. We focus on the Gaussian FP and ${}_6P_+$ which gives

$$\text{GFP}_6 \quad {}_6\theta_0 = -2 \text{ for } {}_6\mathbf{v}_0^+ = (1, 0), \quad (115)$$

$$\text{GFP}_6 \quad {}_6\theta_0 = -2 \text{ for } {}_6\mathbf{v}_0^- = \left(-\frac{3\pi^2}{\sqrt{5}}, 1\right), \quad (116)$$

$${}_6P_+ \quad {}_6\theta_1 \sim 4.859 \text{ for } {}_6\mathbf{v}_1 \sim (-185.549, 1), \quad (117)$$

$${}_6P_+ \quad {}_6\theta_2 \sim -0.9 \text{ for } {}_6\mathbf{v}_2 \sim 10^{-1}(31.289, 1). \quad (118)$$

The GFP has one relevant (mass) direction, and one marginally relevant direction for positive λ , which signals asymptotic freedom. Notice that for negative λ we do not expect the theory to be non-perturbatively well-defined. On the other hand, the NGFP has a relevant and irrelevant direction and share a similar structure as the Wilson-Fisher FP. The analysis of perturbations around the fixed points in this case, then, leads to the phase diagram and RG flow presented in Fig.9. Same conclusions discussed so far hold again in the present rank 6.

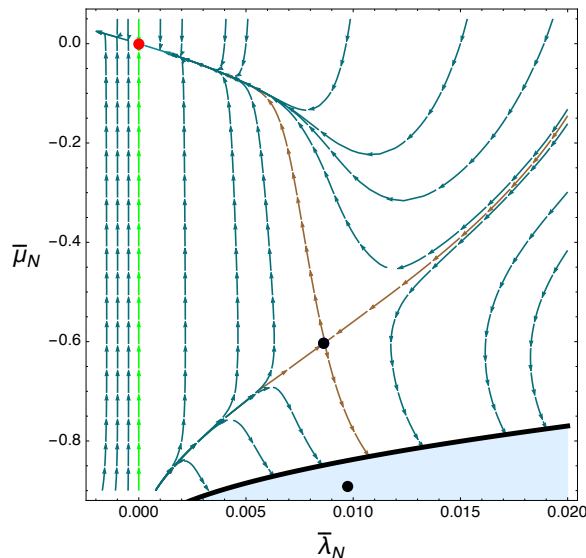


FIG. 9. Flow of the gauged model at rank 6.

After the following change of normalisation $\lambda \rightarrow 2\lambda$, the NGFP ${}_6P_+$, its critical exponents and those of the GFP match with the results in rank $d = 6$ in the large mode limit of [40]. The RG flow lines are also very similar. Interestingly, at least for this model at rank 6, this coincidence means that the large radius sphere limit of the TGFT corresponds to our thermodynamic limit with our particular choice of scaling the coupling including both IR cut-off-scaling and lattice spacing scaling. In fact, we expect this to be true more generally (for instance at any rank d or for any background of the fields).

As pointed out in [40], the presence of both an attractive UV fixed point and an IR Wilson-Fisher fixed point seems to be a general feature of TGFT's. While many other (local) Quantum Field Theories present just one of these results (this is the case of QCD for just asymptotic freedom and of scalar field theory for the IR fixed point), the non-local models that we studied appear to always have a well defined behaviour in both the limits. Moreover, there is another important property which might be interesting and fertile for future developments. All the models that we studied also present a second IR fixed point lying beyond the singularity of the flow. Even if we said that the presence of the anomalous dimension as a parameter in our effective action generates a divergence which prevents us from trusting in the flow across itself, we should remember that far from the infinite values of the flow the computation is probably correct. In other words, given initial conditions in the sector connected with the origin, we are not able to integrate the RG equation beyond the singularity but, had we given initial conditions in the other sector, the situation would be the opposite. Even if we cannot reconnect the flows over all the space of couplings, there are hints that other fixed points could arise and, with them, there is the possibility to find new (non-trivial) UV attractors. If this is confirmed by further investigations, TGFTs would also show asymptotic safety in the UV, in some regions of parameter space, and for specific models at least. If reproduced for 4d gravity models with more quantum geometric structure, this result would be in agreement with the hypothesis of asymptotic safety proposed by Weinberg and Reuter for quantum gravity theories [63]. However, it is not immediate to match TGFT results of this type with the asymptotic safety programme for quantum gravity, since this is based directly on quantum Einstein gravity, thus quantum field theories on spacetime involving directly a metric field, while TGFTs aim to be models of the microscopic constituents of spacetime and geometry itself. Still, it may taken to suggest a nice convergence of results from different directions.

V. CONCLUSION

We have undertaken the Functional Renormalisation Group analysis of two classes of Tensorial Group Field Theories, as a further application of the formalism first studied in [34].

The models are defined on the non-compact group manifold \mathbb{R} and for arbitrary tensor rank. They are endowed with melonic combinatorial interactions and distinguished from the presence (or absence) of a projection on the gauge invariant dynamics under the diagonal group action on the field arguments.

Both classes of models are simplified with respect to full-fledged TGFT models for quantum gravity, usually based

on the group manifolds $SU(2)$ or $SL(2, \mathbb{C})$, and characterised by additional condition on the dynamics, in addition to the gauge invariance models. However, they may capture many of their relevant features, and they are in any case of great interest from a more technical/mathematical point of view, and the FRG analysis is a further step towards controlling and understanding this new type of quantum field theories. More generally, any GFT defines a sum over cellular complexes, which can be interpreted as a discrete definition of the covariant path integral for quantum gravity (with the details of the interpretation depending of course on the details of the amplitudes of the model), of the same type as those defining the dynamical triangulations approach to quantum gravity. The FRG analysis has the main objective of probing their continuum limit and phase structure, which would be, for quantum gravity models, a continuum limit for the pre-geometric, discrete and quantum building blocks of spacetime. The search for a continuum geometric phase governed by a general relativistic dynamics is in fact the main outstanding open issue of these quantum gravity theories.

At a more technical level, the specific aim of our study was to obtain a picture of the fixed points and phase diagram, while enlightening the peculiarities coming from the non-compactness of the underlying group manifold, and thus comparing these results to previous work on TGFTs based on abelian compact groups [34, 40].

The main new issue posed by the non-compactness of the group manifold is the presence of IR divergences in the expansion of the Wetterich equation, which cannot be dealt with in the same way in which one removes simple infinite volume factors in local field theories, due to the particular combinatorics of TGFT interaction terms. We have shown, generalising the previous work [39], how to regularise, first, and then remove these divergences using the appropriate thermodynamic limit. In particular, a comparison with [40] and the verified matching of critical exponents and scaling dimensions, suggests a new concept of scaling dimension for this class of theories. While in the previous work the dimensional analysis leading to scaling dimensions was based on a perturbative approach and on the analysis of n -loops greens functions, at this non-perturbative level we find more appropriate to rely on the order of divergences that need to be regularised to make the theory consistent in the non-compact limit. In this limit, all the models we study define a well-posed autonomous system of RG equations for the coupling constants, we then proceed to solve numerically for various interesting values of the rank, in a simple truncation of the effective action.

In this simple truncation, and for all models considered, we identify UV and IR fixed points, study the perturbations around them, and obtain the corresponding phase diagram. In all these models, we find strong indications of: 1) asymptotic freedom in the UV; 2) a number of non-Gaussian fixed points in the IR; 3) a phase transition similar to the Wilson-Fisher type, between a symmetric and a broken (or condensate) phase with a non-zero expectation value of the TGFT field operator.

The first point is interesting because it confirms, by different means, the apparently generic asymptotic freedom of TGFT models, due to the dominance of wave function renormalisation over coupling constant renormalisation [29]. The last point, on the other hand, is important because phase transitions (in particular, of condensation type) have been suggested to mark the emergence of spacetime and geometry in GFT models of 4d quantum gravity [12, 64], and because GFT condensate states have in fact been used to extract effective cosmological dynamics directly from the microscopic GFT quantum dynamics [44].

However, more work is certainly needed to further corroborate these findings.

Even for this simple class of TGFT models, one would need to improve the truncation scheme to include more terms in the effective action entering the Wetterich equation. And, concerning the study of the phase transition, a clear understanding of the different phases require at least solving the equations of motion (thus a mean field analysis), which is highly non-trivial due to the combinatorial structure of the TGFT interactions and the integro-differential nature of the equations, and a change of parametrisation for the effective potential (see the discussion in [34]).

And of course, we need to proceed towards the FRG analysis of more involved models, investigating how different groups and more involved forms of interaction kernels affect the results, and especially towards models with a more complete quantum geometric interpretation, and stronger links with simplicial quantum gravity and loop quantum gravity. The road ahead is long but promising.

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Appendix A: Evaluation of β -functions in rank d

In this appendix, we provide the detailed calculation of the β equations and emphasise its particularities. Note that, this computation of the β -functions is performed in the regularised framework and only, at the end, we take the

thermodynamic limit. The system of equations that we obtain is an autonomous system in a continuous non-compact space.

Notations. Given the regularization prescription introduced in section III C, we set the notation $\delta_{D^*}(\mathbf{p}, \mathbf{q}) = \delta(\mathbf{p} - \mathbf{q})$ not to be confused with the continuous Dirac delta that we do not use in this appendix. We also define \mathcal{D} to be the one dimensional lattice, that is, the domain of a single component of objects in D^* . We have $D^* = \mathcal{D}^{\times d}$ so that:

$$l \sum_{p_i} = \int_{\mathcal{D}} dp_i. \quad (\text{A.1})$$

A change of notation helps during the calculation:

$$\mathbf{q} = (q_1, q_2, \dots, q_d) \Rightarrow q_1 := q_1; \quad \mathbf{q}_1^{(d-1)} := (q_2, q_3, \dots, q_d); \quad q_1^{(d-1)} := \sqrt{q_2^2 + q_3^2 + \dots, q_d^2},$$

for a generic d -dimensional momentum \mathbf{q} . When there is no possible confusion, we will simply forget the subscript 1 of $\mathbf{q}_1^{(d-1)}$ and $q_1^{(d-1)}$, and use $\mathbf{q}^{(d-1)}$ and $q^{(d-1)}$, respectively.

Let us recall the second variation of the effective action (27) in these new notations:

$$\begin{aligned} \Gamma_k^{(2)} &= (Z_k \sum_s p_s^2 + \mu_k) \delta(\mathbf{p} - \mathbf{p}') \\ &+ \lambda_k \left[\int_{\mathcal{D}^{\times d-1}} dq_2 \dots dq_d \varphi_{p'_1 q_2 \dots q_d} \bar{\varphi}_{p_1 q_2 \dots q_d} \prod_{i=2}^d \delta(p_i - p'_i) \right. \\ &\left. + \int_{\mathcal{D}} dq_1 \varphi_{q_1 p'_2 \dots p'_d} \bar{\varphi}_{q_1 p_2 \dots p_d} \delta(p_1 - p'_1) + \text{sym}\{1, 2, \dots, d\} \right] \\ &= (Z_k \sum_s p_s^2 + \mu_k) \delta(\mathbf{p} - \mathbf{p}') + F_k(\mathbf{p}, \mathbf{p}') \end{aligned}$$

and choose a regulator of the form (21) where θ is now replaced by $\Theta(f(\mathbf{p}))$ the discrete step function. This implies:

$$\partial_t R_k = \delta(\mathbf{p} - \mathbf{p}') \Theta(k^2 - \sum_s p_s^2) [\partial_t Z_k(k^2 - \sum_s p_s^2) + Z_k 2k^2].$$

Defining $P_k(\mathbf{p}, \mathbf{p}')$ like (28), with appropriate replacements, we expand and truncate the Wetterich equation as (30). The zeroth order of the previous expansion is the vacuum term and does not provide us any useful information. On the other hand, the first and the second order will provide us with the flow of the kinetic (φ^2 -) and interaction (φ^4 -) couplings, respectively, namely, the β -functions for the couplings μ_k , Z_k and λ_k .

1. φ^2 -terms

To compute the flow of couplings of the quadratic terms of Γ_k , in other words, the β -functions for μ_k and Z_k , we focus on the first order of (30). To have more compact notations, let us introduce the first convolution appearing in the expansion:

$$\begin{aligned} \tilde{\partial}_t R_k(\mathbf{p}, \mathbf{p}'') &= \int_{D^*} d\mathbf{p}' \partial_t R_k(\mathbf{p}, \mathbf{p}') (P_k)^{-1}(\mathbf{p}', \mathbf{p}'') \\ &= \int_{D^*} d\mathbf{p}' \delta(\mathbf{p} - \mathbf{p}') \delta(\mathbf{p}' - \mathbf{p}'') \Theta(k^2 - \sum_s p_s^2) \frac{\partial_t Z_k(k^2 - \sum_s p_s^2) + 2k^2 Z_k}{Z_k(k^2 - \sum_s p_s'^2) \Theta(k^2 - \sum_s p_s'^2) + Z_k \sum_s p_s'^2 + \mu_k} \\ &= \delta(\mathbf{p} - \mathbf{p}'') \Theta(k^2 - \sum_s p_s^2) \frac{\partial_t Z_k(k^2 - \sum_s p_s^2) + 2k^2 Z_k}{(Z_k k^2 + \mu_k)}, \end{aligned}$$

where we used the fact that, after integration, the two Θ 's appearing in the expression are redundant.

Thus, calling $(I)_W$ the first order of the Wetterich equation, we write

$$\begin{aligned} -(I)_W &= \overline{\text{Tr}}[\tilde{\partial}_t R_k \cdot F_k \cdot (P_k)^{-1}] = \int_{D^* \times 2} d\mathbf{p} d\mathbf{p}' \tilde{\partial}_t R_k(\mathbf{p}, \mathbf{p}') \int_{D^*} d\mathbf{q} F_k(\mathbf{p}', \mathbf{q}) (P_k)^{-1}(\mathbf{q}, \mathbf{p}) \\ &= \int_{D^*} d\mathbf{p} \Theta(k^2 - \sum_s p_s^2) \frac{\partial_t Z_k(k^2 - \sum_s p_s^2) + 2k^2 Z_k}{(Z_k k^2 + \mu_k)^2} F_k(\mathbf{p}, \mathbf{p}). \end{aligned}$$

To simplify the computation, we split the integral in two pieces, namely:

$$A = \frac{\partial_t Z_k}{(Z_k k^2 + \mu_k)^2} \int_{D^*} d\mathbf{p} \, \Theta(k^2 - \sum_s p_s^2) \left(\sum_s p_s^2 \right) F_k(\mathbf{p}, \mathbf{p}),$$

$$B = \frac{k^2(2 + \partial_t) Z_k}{(Z_k k^2 + \mu_k)^2} \int_{D^*} d\mathbf{p} \, \Theta(k^2 - \sum_s p_s^2) F_k(\mathbf{p}, \mathbf{p}), \quad (\text{A.2})$$

having $(I)_W = A - B$. Let us treat the first term and recall that $\delta_{\mathcal{D}}(0) = \delta(0) = \frac{1}{l}$:

$$\begin{aligned} A &= \frac{\lambda_k \partial_t Z_k}{(Z_k k^2 + \mu_k)^2} \int_{D^*} d\mathbf{p} \, \Theta(k^2 - \sum_s p_s^2) \left(\sum_s p_s^2 \right) \\ &\quad \times \left[\frac{1}{l^{d-1}} \int_{\mathcal{D}^{\times d-1}} dq_2 \dots dq_d |\varphi_{p_1 q_2 \dots q_d}|^2 + \frac{1}{l} \int_{\mathcal{D}} dq_1 |\varphi_{q_1 p_2 \dots p_d}|^2 + \text{sym}\{1, 2, \dots, d\} \right] \\ &= \frac{\lambda_k \partial_t Z_k}{(Z_k k^2 + \mu_k)^2} \times \\ &\quad \left\{ \frac{1}{l^{d-1}} \int_{D^*} dp_1 dq_2 \dots dq_d |\varphi_{p_1 q_2 \dots q_d}|^2 \int_{\mathcal{D}^{\times d-1}} dp_2 \dots dp_d \Theta[(k^2 - p_1^2) - \sum_{i=2}^d p_i^2] \left[\sum_{i=2}^d p_i^2 + p_1^2 \right] \right. \\ &\quad \left. + \frac{1}{l} \int_{D^*} dq_1 dp_2 \dots dp_d |\varphi_{q_1 p_2 \dots p_d}|^2 \int_{\mathcal{D}} dp_1 \Theta[(k^2 - \sum_{i=2}^d p_i^2) - p_1^2] \left[\sum_{i=2}^d p_i^2 + p_1^2 \right] \right\} \\ &\quad + \text{sym}\{1, 2, \dots, d\}. \end{aligned}$$

Now we perform the continuum limit $l \rightarrow \infty$ and this corresponds to:

$$\int_{\mathcal{D}} \longrightarrow \int_{\mathbb{R}}, \quad \Theta \longrightarrow \theta. \quad (\text{A.3})$$

The negative powers of l appearing in the expressions keep track of the former IR divergences of the continuous model. Extracting an l dependence from the couplings, we will address them at the end. In order to simplify the notation, we drop the limit symbol $\lim_{l \rightarrow \infty}$ and get

$$\begin{aligned} A &= \frac{\lambda_k \partial_t Z_k}{(Z_k k^2 + \mu_k)^2} \\ &\quad \times \left\{ \frac{1}{l} \int_{\mathbb{R}^d} dq_1 dp_2 \dots dp_d \theta(k^2 - \sum_{i=2}^d p_i^2) |\varphi_{q_1 p_2 \dots p_d}|^2 \int_{-\sqrt{k^2 - \sum_{i=2}^d p_i^2}}^{\sqrt{k^2 - \sum_{i=2}^d p_i^2}} dp_1 \left[\sum_{i=2}^d p_i^2 + p_1^2 \right] \right. \\ &\quad \left. + \frac{1}{l^{d-1}} \int_{\mathbb{R}^d} dp_1 dq_2 \dots dq_d \theta(k^2 - p_1^2) |\varphi_{p_1 q_2 \dots q_d}|^2 \int d\Omega_{d-1} \int_0^{\sqrt{k^2 - p_1^2}} dr r^{d-2} [r^2 + p_1^2] \right\} \\ &\quad + \text{sym}\{1, 2, \dots, d\} \\ &= \frac{\lambda_k \partial_t Z_k}{(Z_k k^2 + \mu_k)^2} \left\{ \frac{1}{l} \int_{\mathbb{R}^d} dq_1 dp_2 \dots dp_d \theta(k^2 - \sum_{i=2}^d p_i^2) \right. \\ &\quad \times \left[2(\sum_{i=2}^d p_i^2) \sqrt{k^2 - \sum_{i=2}^d p_i^2} + \frac{2}{3}(k^2 - \sum_{i=2}^d p_i^2)^{3/2} \right] |\varphi_{q_1 p_2 \dots p_d}|^2 \\ &\quad \left. + \frac{1}{l^{d-1}} \int_{\mathbb{R}^d} dp_1 dq_2 \dots dq_d \theta(k^2 - p_1^2) \left[\frac{(k^2 - p_1^2)^{\frac{d+1}{2}}}{d+1} + \frac{p_1^2}{d-1} (k^2 - p_1^2)^{\frac{d-1}{2}} \right] \Omega_{d-1} |\varphi_{p_1 q_2 \dots q_d}|^2 \right\} \\ &\quad + \text{sym}\{1, 2, \dots, d\}, \end{aligned}$$

where in the first passage we changed variable to the $d-1$ dimensional spherical coordinates and introduced the following notation:

$$\Omega_d = \int d\Omega_d = \prod_{i=1}^{d-2} \left[\int_0^\pi d\alpha_i \sin^{d-1-i}(\alpha_i) \right] \int_0^{2\pi} d\alpha_{d-1} = \frac{2\pi^{d/2}}{\Gamma_E(\frac{d}{2})}, \quad (\text{A.4})$$

with Γ_E the Euler gamma function. Expanding the term B , we find:

$$\begin{aligned}
B &= \lambda_k \frac{k^2(2 + \partial_t)Z_k}{(Z_k k^2 + \mu_k)^2} \\
&\times \int_{D^*} d\mathbf{p} \Theta(k^2 - \sum_s p_s^2) \left[\frac{1}{l^{d-1}} \int_{\mathcal{D} \times d-1} dq_2 \dots dq_d |\varphi_{p_1 q_2 \dots q_d}|^2 + \frac{1}{l} \int_{\mathcal{D}} dq_1 |\varphi_{q_1 p_2 \dots p_d}|^2 \right] \\
&+ \text{sym}\{1, 2, \dots, d\},
\end{aligned} \tag{A.5}$$

which, in the limit, gives

$$\begin{aligned}
B &= \lambda_k \frac{k^2(2 + \partial_t)Z_k}{(Z_k k^2 + \mu_k)^2} \left\{ \frac{1}{l^{d-1}} \int_{\mathbb{R}^d} dp_1 dq_2 \dots dq_d \theta(k^2 - p_1^2) |\varphi_{p_1 q_2 \dots q_d}|^2 \Omega_{d-1} \int_0^{\sqrt{k^2 - p_1^2}} dr r^{d-2} \right. \\
&+ \frac{1}{l} \int_{\mathbb{R}^d} dq_1 dp_2 \dots dp_d \theta(k^2 - \Sigma_{s=2}^d p_s^2) |\varphi_{q_1 p_2 \dots p_d}|^2 \int_{-\sqrt{k^2 - \Sigma_{s=2}^d p_s^2}}^{\sqrt{k^2 - \Sigma_{s=2}^d p_s^2}} dp_1 \Big\} \\
&+ \text{sym}\{1, 2, \dots, d\} \\
&= \lambda_k \frac{k^2(2 + \partial_t)Z_k}{(Z_k k^2 + \mu_k)^2} \left\{ \frac{1}{l^{d-1}} \int_{\mathbb{R}^d} dp_1 dq_2 \dots dq_d \theta(k^2 - p_1^2) \Omega_{d-1} \frac{(k^2 - p_1^2)^{\frac{d-1}{2}}}{d-1} |\varphi_{p_1 q_2 \dots q_d}|^2 \right. \\
&+ \frac{2}{l} \int_{\mathbb{R}^d} dq_1 dp_2 \dots dp_d \theta(k^2 - \Sigma_{s=2}^d p_s^2) \sqrt{k^2 - \Sigma_{s=2}^d p_s^2} |\varphi_{q_1 p_2 \dots p_d}|^2 \Big\} \\
&+ \text{sym}\{1, 2, \dots, d\}.
\end{aligned}$$

β -functions. To find the β -functions of the coupling constants, we rely on the fact that the l.h.s. of (30) is of the form:

$$\partial_t \Gamma_{kin} = \int d\mathbf{p} |\varphi(\mathbf{p})|^2 \left(\beta(Z_k) \sum_s p_s^2 + \beta(\mu_k) \right).$$

In fact, this allows us to identify the β -functions with the coefficients of an expansion in powers of the field momenta of the integrands in A and B , up to an $o(p^3)$. Respectively, the terms with momenta of order p_i^2 convoluted with the fields $\varphi_{\dots p_i, \dots}$ will contribute to the flow of the wave function renormalisation, while the zeroth order will be proportional to the scaling of the mass. All remaining terms, falling out of the truncation, must be discarded. Hence, we have, for $d \geq 3$,

$$\begin{aligned}
A &\simeq \frac{\lambda_k \partial_t Z_k}{(Z_k k^2 + \mu_k)^2} \left\{ \frac{1}{l} \int dq_1 dp_2 \dots dp_d \left[\frac{2}{3} k^3 + k \left(\Sigma_{s=2}^d p_s^2 \right) \right] |\varphi_{q_1 p_2 \dots p_d}|^2 \right. \\
&+ \frac{1}{l^{d-1}} \int dp_1 dq_2 \dots dq_d \Omega_{d-1} \left[\frac{k^{d+1}}{d+1} + \left(\frac{1}{d-1} - \frac{1}{2} \right) k^{d-1} p_1^2 \right] |\varphi_{p_1 q_2 \dots q_d}|^2 \Big\} \\
&+ \text{sym}\{1, 2, \dots, d\}.
\end{aligned}$$

For the B terms, one finds:

$$\begin{aligned}
B &\simeq \lambda_k \frac{k^2(2 + \partial_t)Z_k}{(Z_k k^2 + \mu_k)^2} \left\{ \frac{1}{l^{d-1}} \int dp_1 dq_2 \dots dq_d \Omega_{d-1} \left[\frac{k^{d-1}}{d-1} - \frac{k^{d-3}}{2} p_1^2 \right] |\varphi_{p_1 q_2 \dots q_d}|^2 \right. \\
&+ \frac{2}{l} \int dq_1 dp_2 \dots dp_d \left[k - \frac{1}{2k} \left(\Sigma_{s=2}^d p_s^2 \right) \right] |\varphi_{q_1 p_2 \dots p_d}|^2 \Big\} \\
&+ \text{sym}\{1, 2, \dots, d\}.
\end{aligned} \tag{A.6}$$

Now, we concentrate on the coloured symmetric terms. Note that the procedure and result of the above integrals will not change for each coloured term in $\text{sym}\{\cdot\}$, up to a simple relabelling. Thus, collecting all terms, we obtain an

expression of the form:

$$\begin{aligned}\partial_t \Gamma_{kin} &= \int dp_1 \dots dp_d |\varphi_{p_1 \dots p_d}|^2 \sum_{j=1}^d \left[f(k) + g(k)p_j^2 + h(k) \left(\sum_{i=1}^{j-1} p_i^2 + \sum_{i=j+1}^d p_i^2 \right) \right] \\ &= \int dp_1 \dots dp_d |\varphi_{p_1 \dots p_d}|^2 \left\{ d f(k) + \left[g(k) + (d-1)h(k) \right] \sum_{i=1}^d p_i^2 \right\}.\end{aligned}\quad (\text{A.7})$$

This, by comparison between the two sides of the equation, leads to the following dimensionful β -functions for the parameters Z_k and μ_k :

$$\begin{aligned}\beta(Z_k) &= \frac{\lambda_k}{(Z_k k^2 + \mu_k)^2} \left\{ \partial_t Z_k \left[2(d-1) \frac{k}{l} + \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+1}{2}\right)} \frac{k^{d-1}}{l^{d-1}} \right] + 2Z_k \left[(d-1) \frac{k}{l} + \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d-1}{2}\right)} \frac{k^{d-1}}{l^{d-1}} \right] \right\} \\ \beta(\mu_k) &= -\frac{d \lambda_k}{(Z_k k^2 + \mu_k)^2} \left\{ \partial_t Z_k \left[\frac{4}{3} \frac{k^3}{l} + \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+3}{2}\right)} \frac{k^{d+1}}{l^{d-1}} \right] + 2Z_k \left[2 \frac{k^3}{l} + \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+1}{2}\right)} \frac{k^{d+1}}{l^{d-1}} \right] \right\}.\end{aligned}\quad (\text{A.8})$$

Already at this level, one realises that each β -function does not have homogeneous scaling in k and dimensions in l . This feature clearly comes from the pattern of the convolution of the interaction which is specific to TGFTs.

2. φ^4 -terms

The second order $(II)_W$ of (30) will provide the β -function for λ_k , which completes the set of β -functions of the model. Defining, R'_k and P'_k such that

$$\begin{aligned}R_k(\mathbf{p}, \mathbf{p}') &= R'_k(\mathbf{p}) \Theta(k^2 - \sum_s p_s^2) \delta(\mathbf{p} - \mathbf{p}'), \\ P_k(\mathbf{p}, \mathbf{p}') &= P'_k(\mathbf{p}) \delta(\mathbf{p} - \mathbf{p}'),\end{aligned}\quad (\text{A.9})$$

the terms of interest take the form:

$$\begin{aligned}(II)_W &= \text{Tr}[\partial_t R_k \cdot (P_k)^{-1} \cdot F_k \cdot (P_k)^{-1} \cdot F_k \cdot (P_k)^{-1}] \\ &= \int_{D^{* \times 5}} d\mathbf{p} d\mathbf{p}' d\mathbf{p}'' d\mathbf{q} d\mathbf{q}' \partial_t R'_k(\mathbf{p}) \Theta(k^2 - \sum_s p_s^2) \delta(\mathbf{p} - \mathbf{p}') (P'_k)^{-1}(\mathbf{p}') \delta(\mathbf{p}' - \mathbf{p}'') \\ &\quad \times F_k(\mathbf{p}'', \mathbf{q}) (P'_k)^{-1}(\mathbf{q}) \delta(\mathbf{q} - \mathbf{q}') F_k(\mathbf{q}', \mathbf{p}) (P'_k)^{-1}(\mathbf{p}) \\ &= \int_{D^*} d\mathbf{p} \partial_t R'_k(\mathbf{p}) \Theta(k^2 - \sum_s p_s^2) (P'_k)^{-1}(\mathbf{p}) \int_{D^*} d\mathbf{q} F_k(\mathbf{p}, \mathbf{q}) (P'_k)^{-1}(\mathbf{q}) F_k(\mathbf{q}, \mathbf{p}) (P'_k)^{-1}(\mathbf{p}).\end{aligned}\quad (\text{A.10})$$

We focus on the intermediate convolution $F_k \cdot P_k^{-1} \cdot F_k$ which expands as:

$$\begin{aligned}(F_k \cdot P_k^{-1} \cdot F_k)(\mathbf{p}, \mathbf{p}) &= \lambda_k^2 \int_{D^*} d\mathbf{q} F(\mathbf{p}, \mathbf{q}) (P'_k)^{-1}(\mathbf{q}) F_k(\mathbf{q}, \mathbf{p}) \\ &= \lambda_k^2 \int_{D^*} dq_1 \dots dq_d \left[\int_{\mathcal{D}} dm_1 \varphi_{m_1 p_2 \dots p_d} \bar{\varphi}_{m_1 q_2 \dots q_d} \delta(p_1 - q_1) \right. \\ &\quad \left. + \int_{\mathcal{D} \times d-1} dm_2 \dots dm_d \varphi_{p_1 m_2 \dots m_d} \bar{\varphi}_{q_1 m_2 \dots m_d} \prod_{i=2}^d \delta(p_i - q_i) + \text{sym}\{1, 2, \dots, d\} \right] \\ &\quad (P'_k)^{-1}(\mathbf{q}) \left[\int_{\mathcal{D}} dm'_1 \varphi_{m'_1 q_2 \dots q_d} \bar{\varphi}_{m'_1 p_2 \dots p_d} \delta(p_1 - q_1) \right. \\ &\quad \left. + \int_{\mathcal{D} \times d-1} dm'_2 \dots dm'_d \varphi_{q_1 m'_2 \dots m'_d} \bar{\varphi}_{p_1 m'_2 \dots m'_d} \prod_{i=2}^d \delta(p_i - q_i) + \text{sym}\{1, 2, \dots, d\} \right].\end{aligned}$$

At this level, the product of coloured symmetric terms generates a list of terms (among which cross terms) that we must all carefully analyse. First, we deal with the case when the product involves two terms of the same colour, then

we will treat the cross-coloured case. Below, we further specialise the study to the product of terms of colour 1 and, then on the cross term 1-2 in the above expansion. We refer to the first type of term as $(F_k \cdot P_k^{-1} \cdot F_k)(\mathbf{p}, \mathbf{p})|_{1,1}$ and to the overall contribution after tracing over remaining indices as $(II)_W|_{1,1}$ (respectively, the symbol $|_{1,2}$ will stand for the cross term product of the colours 1 and 2). This evaluation is, of course, without loss of generality because one can quickly infer the result for all remaining products. All these contributions, at the end, must be summed.

We have

$$\begin{aligned}
& (F_k \cdot P_k^{-1} \cdot F_k)(\mathbf{p}, \mathbf{p})|_{1,1} = \\
& \lambda_k^2 \int_{D^*} dq_1 \dots dq_d \int_{\mathcal{D}} dm_1 \varphi_{m_1 p_2 \dots p_d} \bar{\varphi}_{m_1 q_2 \dots q_d} \delta(p_1 - q_1) (P'_k)^{-1}(\mathbf{q}) \\
& \times \int_{\mathcal{D}^{\times d-1}} dm'_2 \dots dm'_d \varphi_{q_1 m'_2 \dots m'_d} \bar{\varphi}_{p_1 m'_2 \dots m'_d} \prod_{i=2}^d \delta(p_i - q_i) \\
& + \lambda_k^2 \int_{D^*} dq_1 \dots dq_d \int_{\mathcal{D}^{\times d-1}} dm_2 \dots dm_d \varphi_{p_1 m_2 \dots m_d} \bar{\varphi}_{q_1 m_2 \dots m_d} \delta(p_i - q_i) (P'_k)^{-1}(\mathbf{q}) \\
& \times \int_{\mathcal{D}} dm'_1 \varphi_{m'_1 q_2 \dots q_d} \bar{\varphi}_{m'_1 p_2 \dots p_d} \delta(p_1 - q_1) \\
& + \lambda_k^2 \int_{D^*} dq_1 \dots dq_d \int_{\mathcal{D}} dm_1 \varphi_{m_1 p_2 \dots p_d} \bar{\varphi}_{m_1 q_2 \dots q_d} \delta(p_1 - q_1) (P'_k)^{-1}(\mathbf{q}) \\
& \times \int_{\mathcal{D}} dm'_1 \varphi_{m'_1 q_2 \dots q_d} \bar{\varphi}_{m'_1 p_2 \dots p_d} \delta(p_1 - q_1) \\
& + \lambda_k^2 \int_{D^*} dq_1 \dots dq_d \int_{\mathcal{D}^{\times d-1}} dm_2 \dots dm_d \varphi_{p_1 m_2 \dots m_d} \bar{\varphi}_{q_1 m_2 \dots m_d} \prod_{i=2}^d \delta(p_i - q_i) (P'_k)^{-1}(\mathbf{q}) \\
& \times \int_{\mathcal{D}^{\times d-1}} dm'_2 \dots dm'_d \varphi_{q_1 m'_2 \dots m'_d} \bar{\varphi}_{p_1 m'_2 \dots m'_d} \prod_{i=2}^d \delta(p_i - q_i).
\end{aligned}$$

The first two terms, once that the δ 's in \mathbf{q} are integrated out, become proportional to the product of two square modulus of the fields, thus they represent disconnected interactions. They can be discarded for the same reasons invoked above. As a remainder, we get:

$$\begin{aligned}
& (F_k \cdot P_k^{-1} \cdot F_k)(\mathbf{p}, \mathbf{p})|_{1,1} \simeq \\
& \frac{\lambda_k^2}{l} \int_{\mathcal{D}^{\times d-1}} dq_2 \dots dq_d \int_{\mathcal{D}^{\times 2}} dm_1 dm'_1 \varphi_{m_1 p_2 \dots p_d} \bar{\varphi}_{m_1 q_2 \dots q_d} \varphi_{m'_1 q_2 \dots q_d} \bar{\varphi}_{m'_1 p_2 \dots p_d} (P'_k)^{-1}(p_1, q_2, \dots, q_d) \\
& + \frac{\lambda_k^2}{l^{d-1}} \int_{\mathcal{D}} dq_1 \int_{\mathcal{D}^{\times 2d-2}} dm_2 \dots dm_d dm'_2 \dots dm'_d \varphi_{p_1 m_2 \dots m_d} \bar{\varphi}_{q_1 m_2 \dots m_d} \varphi_{q_1 m'_2 \dots m'_d} \bar{\varphi}_{p_1 m'_2 \dots m'_d} (P'_k)^{-1}(q_1, p_2, \dots, p_d).
\end{aligned} \tag{A.11}$$

Then, plugging back (A.11) in $(II)_W$ and concentrating on the contribution of this term, one finds:

$$\begin{aligned}
& (II)_W|_{1,1} = \lambda_k^2 \int_{D^*} d\mathbf{p} \Theta(k^2 - \sum_s p_s^2) \frac{[\partial_t Z_k(k^2 - \sum_s p_s^2) + 2k^2 Z_k]}{(Z_k k^2 + \mu_k)^2} \\
& \left\{ \frac{1}{l} \int_{\mathcal{D}^{\times d-1}} dq_2 \dots dq_d \int_{\mathcal{D}^{\times 2}} dm_1 dm'_1 \varphi_{m_1 p_2 \dots p_d} \bar{\varphi}_{m_1 q_2 \dots q_d} \varphi_{m'_1 q_2 \dots q_d} \bar{\varphi}_{m'_1 p_2 \dots p_d} \right. \\
& \left[Z_k(k^2 - p_1^2 - \sum_{i=2}^d q_i^2) \Theta(k^2 - p_1^2 - \sum_{i=2}^d q_i^2) + Z_k(p_1^2 + \sum_{i=2}^d q_i^2) + \mu_k \right]^{-1} \\
& + \frac{1}{l^{d-1}} \int_{\mathcal{D}} dq_1 \int_{\mathcal{D}^{\times 2d-2}} dm_2 \dots dm_d dm'_2 \dots dm'_d \varphi_{p_1 m_2 \dots m_d} \bar{\varphi}_{q_1 m_2 \dots m_d} \varphi_{q_1 m'_2 \dots m'_d} \bar{\varphi}_{p_1 m'_2 \dots m'_d} \\
& \left. \left[Z_k(k^2 - q_1^2 - \sum_{i=2}^d p_i^2) \Theta(k^2 - q_1^2 - \sum_{i=2}^d p_i^2) + Z_k(q_1^2 + \sum_{i=2}^d p_i^2) + \mu_k \right]^{-1} \right\}.
\end{aligned}$$

With the same principle used for evaluation of the β -functions of Z_k and μ_k , any explicit dependence on the $2d$ momenta involved in the four fields in the spectral sums of (A.10) must be discarded. In other words, any term of the form $p_i^\alpha \varphi \dots p_i \dots \bar{\varphi} \dots p_i \dots \cdot (\varphi \bar{\varphi})$ falls out of the truncation. After taking the limit (again we drop the symbol $\lim_{l \rightarrow 0}$),

we expand the expression at zeroth order and get:

$$\begin{aligned}
(II)_W|_{1,1} &\simeq \frac{\lambda_k^2}{l} \int_{\mathbb{R}^{2d}} dm_1 dm'_1 dp_2 \dots dp_d dq_2 \dots dq_d \varphi_{m_1 p_2 \dots p_d} \bar{\varphi}_{m_1 q_2 \dots q_d} \varphi_{m'_1 q_2 \dots q_d} \bar{\varphi}_{m'_1 p_2 \dots p_d} \\
&\times \int_{\mathbb{R}} dp_1 \frac{[\partial_t Z_k (k^2 - p_1^2) + 2k^2 Z_k]}{(Z_k k^2 + \mu_k)^2} \frac{\theta(k^2 - p_1^2)}{Z_k (k^2 - p_1^2) \theta(k^2 - p_1^2) + Z_k p_1^2 + \mu_k} \\
&+ \frac{\lambda_k^2}{l^{d-1}} \int_{\mathbb{R}^{2d}} dp_1 dq_1 dm_2 \dots dm_d dm'_2 \dots dm'_d \varphi_{p_1 m_2 \dots m_d} \bar{\varphi}_{q_1 m_2 \dots m_d} \varphi_{q_1 m'_2 \dots m'_d} \bar{\varphi}_{p_1 m'_2 \dots m'_d} \\
&\times \int_{\mathbb{R}^{d-1}} dp_2 \dots dp_d \frac{[\partial_t Z_k (k^2 - \sum_{i=2}^d p_i^2) + 2k^2 Z_k]}{(Z_k k^2 + \mu_k)^2} \frac{\theta(k^2 - \sum_{i=2}^d p_i^2)}{Z_k (k^2 - \sum_{i=2}^d p_i^2) \theta(k^2 - \sum_{i=2}^d p_i^2) + Z_k (\sum_{i=2}^d p_i^2) + \mu_k} .
\end{aligned}$$

The θ 's turn out to be redundant in both the terms and we can simplify their contributions. Call \mathcal{V}_i the vertex of colour i of the effective interaction. Rather than using the explicit form of that vertex, we will simply use \mathcal{V}_i in the following, when no confusion might arise.

We split the previous terms in two pieces:

$$\begin{aligned}
(II)'_W|_{1,1} &= \\
&\frac{1}{l} \frac{\lambda_k^2 k^2 (2 + \partial_t) Z_k}{(Z_k k^2 + \mu_k)^3} \int dq_2 \dots dq_d dp_2 \dots dp_d dm_1 dm'_1 \varphi_{m_1 p_2 \dots p_d} \bar{\varphi}_{m_1 q_2 \dots q_d} \varphi_{m'_1 q_2 \dots q_d} \bar{\varphi}_{m'_1 p_2 \dots p_d} \int dp_1 \theta(k^2 - p_1^2) \\
&- \frac{1}{l} \frac{\lambda_k^2 \partial_t Z_k}{(Z_k k^2 + \mu_k)^3} \int dq_2 \dots dq_d dp_2 \dots dp_d dm_1 dm'_1 \varphi_{m_1 p_2 \dots p_d} \bar{\varphi}_{m_1 q_2 \dots q_d} \varphi_{m'_1 q_2 \dots q_d} \bar{\varphi}_{m'_1 p_2 \dots p_d} \int dp_1 p_1^2 \theta(k^2 - p_1^2) \\
&= 2 \frac{\lambda_k^2 k^3}{l} \left[\frac{(2 + \partial_t) Z_k}{(Z_k k^2 + \mu_k)^3} - \frac{1}{3} \frac{\partial_t Z_k}{(Z_k k^2 + \mu_k)^3} \right] \mathcal{V}_1 = \frac{2 \lambda_k^2 k^3}{l (Z_k k^2 + \mu_k)^3} \left[2 Z_k + \frac{2}{3} \partial_t Z_k \right] \mathcal{V}_1 .
\end{aligned}$$

The second integral can be computed as:

$$\begin{aligned}
(II)''_W|_{1,1} &= \\
&\frac{1}{l^{d-1}} \frac{\lambda_k^2 k^2 (2 + \partial_t) Z_k}{(Z_k k^2 + \mu_k)^3} \int dp_1 dq_1 dm_2 \dots dm_d dm'_2 \dots dm'_d \varphi_{p_1 m_2 \dots m_d} \bar{\varphi}_{q_1 m_2 \dots m_d} \varphi_{q_1 m'_2 \dots m'_d} \bar{\varphi}_{p_1 m'_2 \dots m'_d} \\
&\times \int dp_2 \dots dp_d \theta(k^2 - \sum_{i=2}^d p_i^2) \\
&- \frac{1}{l^{d-1}} \frac{\lambda_k^2 \partial_t Z_k}{(Z_k k^2 + \mu_k)^3} \int dp_1 dq_1 dm_2 \dots dm_d dm'_2 \dots dm'_d \varphi_{p_1 m_2 \dots m_d} \bar{\varphi}_{q_1 m_2 \dots m_d} \varphi_{q_1 m'_2 \dots m'_d} \bar{\varphi}_{p_1 m'_2 \dots m'_d} \\
&\times \int dp_2 \dots dp_d \left(\sum_{i=2}^d p_i^2 \right) \theta(k^2 - \sum_{i=2}^d p_i^2) \\
&= \int dp_1 dq_1 dm_2 \dots dm_d dm'_2 \dots dm'_d \varphi_{p_1 m_2 \dots m_d} \bar{\varphi}_{q_1 m_2 \dots m_d} \varphi_{q_1 m'_2 \dots m'_d} \bar{\varphi}_{p_1 m'_2 \dots m'_d} \\
&\times \left[\frac{1}{l^{d-1}} \frac{\lambda_k^2 k^2 (2 + \partial_t) Z_k}{(Z_k k^2 + \mu_k)^3} \int d\Omega_{d-1} \int_0^k dr r^{d-2} - \frac{1}{l^{d-1}} \frac{\lambda_k^2 \partial_t Z_k}{(Z_k k^2 + \mu_k)^3} \int d\Omega_{d-1} \int_0^k dr r^d \right] \\
&= \frac{\lambda_k^2}{l^{d-1} (Z_k k^2 + \mu_k)^3} \left[\frac{2k^{d+1} (2 + \partial_t) Z_k \pi^{\frac{d-1}{2}}}{(d-1) \Gamma_E\left(\frac{d-1}{2}\right)} - \frac{2\pi^{\frac{d-1}{2}} k^{d+1} \partial_t Z_k}{(d+1) \Gamma_E\left(\frac{d-1}{2}\right)} \right] \mathcal{V}_1 \\
&= \frac{\lambda_k^2 k^{d+1} \pi^{\frac{d-1}{2}}}{l^{d-1} (Z_k k^2 + \mu_k)^3} \left[\frac{\partial_t Z_k}{\Gamma_E\left(\frac{d+3}{2}\right)} + \frac{2Z_k}{\Gamma_E\left(\frac{d+1}{2}\right)} \right] \mathcal{V}_1 .
\end{aligned}$$

A simple check of the dimensions of these terms and the dimension of the interaction term of the effective action can be given as

$$[(II)'_W] = [(II)''_W] = 2[\lambda] - 4 + 2d + 4[\varphi] ,$$

which, considering that $[\varphi] = -\frac{d+2}{2}$, fixes $[\lambda] = 4$ as expected.

Let us now focus on the cross term given by the product of the contribution of colour 1 and 2:

$$\begin{aligned}
(II)_W|_{1,2} &= \lambda_k^2 \int_{D^* \times 2} d\mathbf{p} d\mathbf{j} \frac{\Theta(k^2 - \sum_s p_s^2)}{(Z_k k^2 + \mu_k)^2} \frac{\partial_t Z_k (k^2 - \sum_s p_s^2) + 2k^2 Z_k}{\Theta(k^2 - \sum_s j_s^2) Z_k (k^2 - \sum_s j_s^2) + Z_k \sum_s j_s^2 + \mu_k} \\
&\left[\int_{\mathcal{D} \times 2} dm_1 dn_2 \varphi_{m_1 j_2 \dots j_d} \bar{\varphi}_{m_1 p_2 \dots p_d} \varphi_{p_1 n_2 p_3 \dots p_d} \bar{\varphi}_{j_1 n_2 j_3 \dots j_d} \delta(p_1 - j_1) \delta(p_2 - j_2) \right. \\
&+ \int_{\mathcal{D}^{2d-2}} dm_2 \dots dm_d dn_1 dn_3 \dots dn_d \varphi_{j_1 m_2 \dots m_d} \bar{\varphi}_{p_1 m_2 \dots m_d} \varphi_{n_1 p_2 n_3 \dots n_d} \bar{\varphi}_{n_1 j_2 n_3 \dots n_d} \\
&\times \delta(p_1 - j_1) \delta(p_2 - j_2) \prod_{i=3}^d \delta^2(p_i - j_i) \\
&+ \int_{D^*} dm_1 dn_1 dn_3 \dots dn_d \varphi_{m_1 j_2 \dots j_d} \bar{\varphi}_{m_1 p_2 \dots p_d} \varphi_{n_1 p_2 n_3 \dots n_d} \bar{\varphi}_{n_1 j_2 n_3 \dots n_d} \delta^2(p_1 - j_1) \prod_{i=3}^d \delta(p_i - j_i) \\
&\left. + \int_{D^*} dm_2 \dots dm_d dn_2 \varphi_{j_1 m_2 \dots m_d} \bar{\varphi}_{p_1 m_2 \dots m_d} \varphi_{p_1 n_2 p_3 \dots p_d} \bar{\varphi}_{j_1 n_2 j_3 \dots j_d} \delta^2(p_2 - j_2) \prod_{i=3}^d \delta(p_i - j_i) \right].
\end{aligned}$$

If we integrate the deltas over the j variables, the second term is again a disconnected 4-point function that we neglect. In rank $d > 3$, the first term falls out of the truncation: it generates a "matrix-like" convolution with two momenta distinguished from the other $d - 2$ labels. However at the boundary value $d = 3$, it will contribute to the flow. We find:

$$\begin{aligned}
(II)_W|_{1,2} &= \delta_{d,3} \frac{\lambda_k^2}{(Z_k k^2 + \mu_k)^2} \int dp_1 dp_2 dp_3 dm_1 dn_2 dj_3 \varphi_{m_1 p_2 j_3} \bar{\varphi}_{m_1 p_2 p_3} \varphi_{p_1 n_2 p_3} \bar{\varphi}_{p_1 n_2 j_3} \\
&\times \frac{\Theta(k^2 - \sum_s p_s^2) [\partial_t Z_k (k^2 - \sum_s p_s^2) + 2k^2 Z_k]}{\Theta(k^2 - p_1^2 - p_2^2 - j_3^2) Z_k (k^2 - p_1^2 - p_2^2 - j_3^2) + Z_k (p_1^2 + p_2^2 + j_3^2) + \mu_k} \\
&+ \frac{\lambda_k^2}{(Z_k k^2 + \mu_k)^2} \frac{1}{l} \int_{D^* \times \mathcal{D}} dp_1 \dots dp_d dj_2 dm_1 dn_3 \dots dn_d \varphi_{m_1 j_2 p_3 \dots p_d} \bar{\varphi}_{m_1 p_2 \dots p_d} \varphi_{n_1 p_2 n_3 \dots n_d} \bar{\varphi}_{n_1 j_2 n_3 \dots n_d} \\
&\times \frac{\Theta(k^2 - \sum_s p_s^2) [\partial_t Z_k (k^2 - \sum_s p_s^2) + 2k^2 Z_k]}{\Theta(k^2 - p_1^2 - j_2^2 - \sum_{i=3}^d p_i^2) Z_k (k^2 - p_1^2 - j_2^2 - \sum_{i=3}^d p_i^2) + Z_k (p_1^2 + j_2^2 + \sum_{i=3}^d p_i^2) + \mu_k} \\
&+ \frac{\lambda_k^2}{(Z_k k^2 + \mu_k)^2} \frac{1}{l} \int_{D^* \times \mathcal{D}} dp_1 \dots dp_d dj_1 dm_2 \dots dm_d dn_2 \varphi_{j_1 m_2 \dots m_d} \bar{\varphi}_{p_1 m_2 \dots m_d} \varphi_{p_1 n_2 p_3 \dots p_d} \bar{\varphi}_{j_1 n_2 p_3 \dots p_d} \\
&\times \frac{\Theta(k^2 - \sum_s p_s^2) [\partial_t Z_k (k^2 - \sum_s p_s^2) + 2k^2 Z_k]}{\Theta(k^2 - j_1^2 - \sum_{i=2}^d p_i^2) Z_k (k^2 - j_1^2 - \sum_{i=2}^d p_i^2) + Z_k (j_1^2 + \sum_{i=2}^d p_i^2) + \mu_k}.
\end{aligned}$$

In the continuum limit, the previous integrals can be evaluated at 0-momentum truncation and the Θ in the denominator, put to 1. One realises that the first term is proportional to $\delta_{d,3} \mathcal{V}_3$, while the second and third terms are (1,2)-coloured symmetric contributions and are proportional to \mathcal{V}_2 and \mathcal{V}_1 , respectively. Casting away the $p_i^2 \varphi_{p_i}^4$ -terms, one infers

$$\begin{aligned}
(II)_W|_{1,2} &\simeq \delta_{d,3} \frac{\lambda_k^2 k^2 (2 + \partial_t) Z_k}{(Z_k k^2 + \mu_k)^3} \mathcal{V}_3 \\
&+ \frac{\lambda_k^2}{(Z_k k^2 + \mu_k)^3} \frac{1}{l} \mathcal{V}_2 \int dp_1 \theta(k^2 - p_1^2) [\partial_t Z_k (k^2 - p_1^2) + 2k^2 Z_k] + \text{sym}\{1 \rightarrow 2\}.
\end{aligned} \tag{A.12}$$

Performing the integrals over the external momenta:

$$(II)_W|_{1,2} = \frac{\lambda_k^2 k^2}{(Z_k k^2 + \mu_k)^3} \left\{ \delta_{d,3} (2 + \partial_t) Z_k \mathcal{V}_3 + \frac{k}{l} \left[-\frac{2}{3} \partial_t Z_k + 2(2 + \partial_t) Z_k \right] (\mathcal{V}_2 + \mathcal{V}_1) \right\}. \tag{A.13}$$

We are in position to sum all contributions. Taking into account the colour symmetry of the vertices, the coefficients obtained from $(II)_W|_{i,i}$ contributes once for each colour i , while the terms coming from the cross terms, i.e. $(II)_W|_{i,j \neq i}$, will appear once for each couple of colours (i, j) , $j \neq i$. Thus the later terms gain a factor $2(d - 1)$. Especially, the term $\delta_{d,3} \mathcal{V}_3$ in (A.13) and the like, at $d = 3$, acquires a factor of 2. Performing these operations, the β -function for λ_k reads:

$$\beta(\lambda_k) = \frac{2\lambda_k^2}{(Z_k k^2 + \mu_k)^3} \left\{ \partial_t Z_k \left[\frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+3}{2}\right)} \frac{k^{d+1}}{l^{d-1}} + \frac{4(2d-1)}{3} \frac{k^3}{l} + 2\delta_{d,3} k^2 \right] \right\}$$

$$+ 2Z_k \left[\frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+1}{2}\right)} \frac{k^{d+1}}{l^{d-1}} + 2(2d-1) \frac{k^3}{l} + 2\delta_{d,3} k^2 \right] \}. \quad (\text{A.14})$$

Dimensionful β -functions. We write the full set of dimensionful β -functions for the model as:

$$\left\{ \begin{aligned} \beta(Z_k) &= \frac{\lambda_k}{(Z_k k^2 + \mu_k)^2} \left\{ \partial_t Z_k \left[2(d-1) \frac{k}{l} - \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+1}{2}\right)} \frac{k^{d-1}}{l^{d-1}} \right] + 2Z_k \left[(d-1) \frac{k}{l} + \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d-1}{2}\right)} \frac{k^{d-1}}{l^{d-1}} \right] \right\} \\ \beta(\mu_k) &= -\frac{d \lambda_k}{(Z_k k^2 + \mu_k)^2} \left\{ \partial_t Z_k \left[\frac{4}{3} \frac{k^3}{l} + \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+3}{2}\right)} \frac{k^{d+1}}{l^{d+1}} \right] + 2Z_k \left[2 \frac{k^3}{l} + \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+1}{2}\right)} \frac{k^{d+1}}{l^{d-1}} \right] \right\} \\ \beta(\lambda_k) &= \frac{2\lambda_k^2}{(Z_k k^2 + \mu_k)^3} \left\{ \partial_t Z_k \left[\frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+3}{2}\right)} \frac{k^{d+1}}{l^{d-1}} + \frac{4(2d-1)}{3} \frac{k^3}{l} + 2\delta_{d,3} k^2 \right] \right. \\ &\quad \left. + 2Z_k \left[\frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+1}{2}\right)} \frac{k^{d+1}}{l^{d-1}} + 2(2d-1) \frac{k^3}{l} + 2\delta_{d,3} k^2 \right] \right\} \end{aligned} \right\} \quad (\text{A.15})$$

which is reported in section III D, (46).

Appendix B: Evaluation of β -functions in the gauge invariant case

The computation of the dimensionful β -functions for the gauge projected model follows roughly the same steps of the calculations of the model without constraints. However, due to the presence of the extra delta's of the gauge projection, the analysis requires, at some point, a different technique. In this appendix, we provide details of the procedure for obtaining the system of the dimensionful RG equations, namely (93) of section IV C, and underline the differences with the previous calculus.

We start by expanding equation (92) of section IV B and focus, first on the φ^2 -terms and then calculate higher order terms.

1. φ^2 -terms

Referring to the conventions introduced at the beginning of section IV B, say (88)–(91), for the scaling of the kinetic term, we have:

$$\begin{aligned} (I^g)_W &= -\text{Tr}[\partial_t R_k \cdot (P_k)^{-1} \cdot F_k \cdot (P_k)^{-1}] \\ &= -\lambda_k \int_{D^*} d\mathbf{p} \, \Theta(k^2 - \Sigma_s p_s^2) \frac{[\partial_t Z_k(k^2 - \Sigma_s p_s^2) + 2k^2 Z_k]}{(Z_k k^2 + \mu_k)^2} \frac{\delta(\Sigma p)}{\delta^2(\Sigma p)} \\ &\quad \times \left[\frac{1}{l^{d-1}} \int_{\mathcal{D} \times d-1} dm_2 \dots dm_d |\varphi_{p_1 m_2 \dots m_d}|^2 \delta^2(\Sigma p) \delta^2(p_1 + m_2 + \dots + m_d) \right. \\ &\quad \left. + \frac{1}{l} \int_{\mathcal{D}} dm_1 |\varphi_{m_1 p_2 \dots p_d}|^2 \delta^2(\Sigma p) \delta^2(m_1 + p_2 + \dots + p_d) + \text{sym}\{1, 2, \dots, d\} \right]. \end{aligned} \quad (\text{B.1})$$

In the same perspective, the square delta's can be reduced as $\delta^2(p) = \delta(p)\delta(0) = \frac{1}{l}\delta(p)$. The second integral in the above expression can be directly computed by integrating over p_1 the $\delta(\Sigma p)$ as

$$\begin{aligned} (I^g)'_W &= -\frac{\lambda_k}{l^2} \int_{D^*} d\mathbf{p} \, \Theta(k^2 - \Sigma_s p_s^2) \frac{[\partial_t Z_k(k^2 - \Sigma_s p_s^2) + 2k^2 Z_k]}{(Z_k k^2 + \mu_k)^2} \delta(\Sigma p) \\ &\quad \times \int_{\mathcal{D}} dm_1 |\varphi_{m_1 p_2 \dots p_d}|^2 \delta(m_1 + p_2 + \dots + p_d) + \text{sym}\{1, 2, \dots, d\} \\ &= -\frac{\lambda_k}{l^2 (Z_k k^2 + \mu_k)^2} \int_{D^*} dm_1 dp_2 \dots dp_d |\varphi_{m_1 p_2 \dots p_d}|^2 \delta(m_1 + p_2 + \dots + p_d) \\ &\quad \times \int_{\mathcal{D}} dp_1 \, \Theta(k^2 - \Sigma_s p_s^2) [\partial_t Z_k(k^2 - \Sigma_s p_s^2) + 2k^2 Z_k] \delta(\Sigma p) + \text{sym}\{1, 2, \dots, d\} \end{aligned}$$

$$\begin{aligned}
&= -\frac{\lambda_k}{l^2(Z_k k^2 + \mu_k)^2} \int_{D^*} d\mathbf{p} |\varphi_{p_1 p_2 \dots p_d}|^2 \delta(p_1 + p_2 + \dots + p_d) \left[dk^2(2 + \partial_t)Z_k - d\partial_t Z_k \Sigma_{s=1}^d p_s^2 \right] \\
&= -\frac{d\lambda_k}{l^2(Z_k k^2 + \mu_k)^2} \int_{D^*} d\mathbf{p} |\varphi_{p_1 p_2 \dots p_d}|^2 \delta(p_1 + p_2 + \dots + p_d) \left[2k^2 Z_k + \partial_t Z_k [k^2 - \Sigma_{s=1}^d p_s^2] \right]. \quad (\text{B.2})
\end{aligned}$$

We discuss now the first term in the brackets in (B.1) that we denote

$$\begin{aligned}
(I^g)''_W &= -\frac{\lambda_k}{l^d(Z_k k^2 + \mu_k)^2} \int_{D^*} dp_1 dm_2 \dots dm_d |\varphi_{p_1 m_2 \dots m_d}|^2 \delta(p_1 + m_2 + \dots + m_d) \\
&\times \int_{\mathcal{D} \times d-1} dp_2 \dots dp_d \Theta(k^2 - \Sigma_s p_s^2) [\partial_t Z_k (k^2 - \Sigma_s p_s^2) + 2k^2 Z_k] \delta(\sum p) + \text{sym}\{1, 2, \dots, d\} \quad (\text{B.3})
\end{aligned}$$

Because of the combinatorial pattern chosen for the interaction, the case $d = 3$ represents again a special situation that we deal with by direct evaluation. We integrate over the third variable, imposing the constraint $p_3 = -(p_1 + p_2)$. The resulting domain of integration of p_2 is known, in the continuous limit, as the θ distribution is non-zero when $-2p_2^2 - 2p_2 p_1 + (k^2 - 2p_1^2) \geq 0$. The boundary of this inequality, solved in p_2 , is given by the roots

$$p_2^\pm = \frac{1}{2} \left(-p_1 \pm \sqrt{2k^2 - 3p_1^2} \right). \quad (\text{B.4})$$

The non-zero values of the Heaviside distribution hold when $p_2 \in [p_2^-, p_2^+]$. There is still a residual constraint over p_1 which has to be imposed in order to keep real the square root appearing in (B.4), that is, $3p_1^2 \leq 2k^2$. Thus, (B.3) becomes

$$\begin{aligned}
(I^g)''_{W;d=3} &= -\frac{\lambda_k}{l^3(Z_k k^2 + \mu_k)^2} \int dp_1 dm_2 dm_3 |\varphi_{p_1 m_2 m_3}|^2 \delta(p_1 + m_2 + m_3) \\
&\times \theta(2k^2 - 3p_1^2) \int_{\frac{1}{2}(-p_1 - \sqrt{2k^2 - 3p_1^2})}^{\frac{1}{2}(-p_1 + \sqrt{2k^2 - 3p_1^2})} dp_2 \{ \partial_t Z_k [k^2 - 2(p_2^2 + p_1^2 + p_2 p_1)] + 2k^2 Z_k \} \\
&+ \text{sym}\{1, 2, \dots, d\} \\
&= -\frac{\lambda_k}{l^3(Z_k k^2 + \mu_k)^2} \int dp_1 dm_2 dm_3 |\varphi_{p_1 m_2 m_3}|^2 \delta(p_1 + m_2 + m_3) \theta(2k^2 - 3p_1^2) \\
&\times \left\{ k^2 \sqrt{2k^2 - 3p_1^2} (2 + \partial_t) Z_k - \frac{3}{2} \sqrt{2k^2 - 3p_1^2} \partial_t Z_k p_1^2 - \frac{1}{6} (2k^2 - 3p_1^2)^{3/2} \partial_t Z_k \right\} \\
&+ \text{sym}\{1, 2, \dots, d\}. \quad (\text{B.5})
\end{aligned}$$

Expanding the last result up to the third order in momenta, one obtains

$$\begin{aligned}
(I^g)''_{W;d=3} &\simeq -\frac{\lambda_k}{l^3(Z_k k^2 + \mu_k)^2} \int dp_1 dm_2 dm_3 |\varphi_{p_1 m_2 m_3}|^2 \delta(p_1 + m_2 + m_3) \\
&\times \left[k^3 \left(\sqrt{2} - \frac{\sqrt{8}}{6} \right) \partial_t Z_k + 2\sqrt{2} k^3 Z_k - \frac{3}{\sqrt{2}} k (1 + \partial_t) Z_k p_1^2 \right] + \text{sym}\{1, 2, \dots, d\} \\
&\simeq -\frac{\lambda_k}{l^3(Z_k k^2 + \mu_k)^2} \int_{D^*} d\mathbf{p} |\varphi_{p_1 p_2 p_3}|^2 \delta(p_1 + p_2 + p_3) \left[2\sqrt{2} dk^3 \left(\frac{1}{3} \partial_t + 1 \right) Z_k - \frac{3}{\sqrt{2}} k (1 + \partial_t) Z_k \left(\sum_{s=1}^d p_s^2 \right) \right], \quad (\text{B.6})
\end{aligned}$$

where in the last line we include the symmetry factors. From this point, and combining it with (B.2) restricted at $d = 3$, we write the β -functions for the couplings μ_k and Z_k as:

$$\begin{aligned}
\beta_{d=3}(Z_k) &= \frac{\lambda_k}{(Z_k k^2 + \mu_k)^2} \left[\frac{3}{\sqrt{2}} \frac{k}{l^3} (1 + \partial_t) Z_k + \frac{3}{l^2} \partial_t Z_k \right]; \\
\beta_{d=3}(\mu_k) &= -\frac{3\lambda_k}{(Z_k k^2 + \mu_k)^2} \left[2\sqrt{2} \frac{k^3}{l^3} \left(1 + \frac{1}{3} \partial_t \right) Z_k + \frac{k^2}{l^2} (2 + \partial_t) Z_k \right]. \quad (\text{B.7})
\end{aligned}$$

At rank $d \geq 4$, the term (B.3) has more integrations to perform and becomes simpler if expressed in spherical coordinates. Considering that the coordinate p_1 is convoluted with the field, we will change basis from (p_2, \dots, p_d) to (r, Ω_{d-1}) . The $\delta(\sum p)$ defines the hyperplane orthogonal to a vector \mathbf{N} of norm $\|\mathbf{N}\| = \sqrt{d}$ and components (in

Cartesian coordinates) $\mathbf{N} = (1, 1, \dots, 1)$. We will call \mathbf{n} the projection of this vector on the subspace orthogonal to p_1 and \mathbf{P} the generic vector on this subspace. In this setting the Dirac delta function becomes

$$\delta(p_1 + \langle \mathbf{P}, \mathbf{n} \rangle) = \delta(p_1 + r\sqrt{d-1} \cos \vartheta) = \frac{\delta\left(\frac{p_1}{r\sqrt{d-1}} + \cos \vartheta\right)}{r\sqrt{d-1}}, \quad (\text{B.8})$$

where ϑ represents the angle between \mathbf{P} and \mathbf{n} . Considering that the scalar product, as the rest of the integrand, is rotational invariant on the $(d-1)$ dimensional space, we can set ϑ to be one of the angles appearing in the spherical measure. After the change of coordinates, equation (B.3) reads

$$\begin{aligned} (I^g)''_{W;d>3} = & -\frac{\lambda_k}{l^d(Z_k k^2 + \mu_k)^2} \int dp_1 dm_2 \dots dm_d \int dr d\Omega_{d-2} \int_0^\pi d\vartheta r^{d-2} \sin^{d-3} \vartheta \frac{\delta\left(\frac{p_1}{r\sqrt{d-1}} + \cos \vartheta\right)}{r\sqrt{d-1}} \\ & \times |\varphi_{p_1 m_2 \dots m_d}|^2 \delta(p_1 + m_2 + \dots + m_d) \theta(k^2 - p_1^2 - r^2) [\partial_t Z_k (k^2 - p_1^2 - r^2) + 2k^2 Z_k] \\ & + \text{sym}\{1, 2, \dots, d\}. \end{aligned} \quad (\text{B.9})$$

We focus on the integral over ϑ and change variable from ϑ to $X = \cos \vartheta$ and get, for $d > 3$,

$$\int_0^\pi d\vartheta \sin^{d-3} \vartheta \delta\left(\frac{p_1}{r\sqrt{d-1}} + \cos \vartheta\right) = \int_{-1}^1 dX (1 - X^2)^{\frac{d-4}{2}} \delta\left(\frac{p_1}{r\sqrt{d-1}} + X\right) = \left[1 - \frac{p_1^2}{r^2(d-1)}\right]^{\frac{d-4}{2}}. \quad (\text{B.10})$$

Substituting (B.10) in (B.9), we get:

$$\begin{aligned} (I^g)''_{W;d>3} = & -\frac{\lambda_k}{l^d(Z_k k^2 + \mu_k)^2} \frac{\Omega_{d-2}}{\sqrt{d-1}} \int dp_1 dm_2 \dots dm_d |\varphi_{p_1 m_2 \dots m_d}|^2 \delta(p_1 + m_2 + \dots + m_d) \\ & \times \theta(k^2 - p_1^2) \left[\theta(d-5) \int_0^{\sqrt{k^2 - p_1^2}} dr r^{d-3} \left[1 - \frac{p_1^2}{(d-1)r^2}\right]^{\frac{d}{2}-2} [\partial_t Z_k (k^2 - p_1^2 - r^2) + 2k^2 Z_k] \right. \\ & \left. + \delta_{d,4} \int_0^{\sqrt{k^2 - p_1^2}} dr r [\partial_t Z_k (k^2 - p_1^2 - r^2) + 2k^2 Z_k] \right] + \text{sym}\{1, 2, \dots, d\}. \end{aligned} \quad (\text{B.11})$$

Expanding the result of the integral over ϑ at the second order in p_1 , we obtain an integral over r of the form:

$$\begin{aligned} (I^g)''_{W;d>3} \simeq & -\frac{\lambda_k}{l^d(Z_k k^2 + \mu_k)^2} \frac{\Omega_{d-2}}{\sqrt{d-1}} \int dp_1 dm_2 \dots dm_d |\varphi_{p_1 m_2 \dots m_d}|^2 \delta(p_1 + m_2 + \dots + m_d) \\ & \times \theta(k^2 - p_1^2) \left[\theta(d-5) \int_0^{\sqrt{k^2 - p_1^2}} dr r^{d-3} \left[1 - \frac{d-4}{2(d-1)r^2} p_1^2\right] [\partial_t Z_k (k^2 - p_1^2 - r^2) + 2k^2 Z_k] \right. \\ & \left. + \delta_{d,4} \int_0^{\sqrt{k^2 - p_1^2}} dr r [\partial_t Z_k (k^2 - p_1^2 - r^2) + 2k^2 Z_k] \right] + \text{sym}\{1, 2, \dots, d\}. \end{aligned} \quad (\text{B.12})$$

Computing the last integral and expanding the result, we expand the r.h.s. of (B.12) to the second order in the momenta convoluted with the fields and this yields:

$$\begin{aligned} (I^g)''_{W;d>3} \simeq & -\frac{\lambda_k}{l^d(Z_k k^2 + \mu_k)^2} \frac{\Omega_{d-2}}{\sqrt{d-1}} \int dp_1 dm_2 \dots dm_d |\varphi_{p_1 m_2 \dots m_d}|^2 \delta(p_1 + m_2 + \dots + m_d) \\ & \times \theta(k^2 - p_1^2) \left\{ \theta(d-5) \left[\frac{2k^d}{d-2} Z_k + \frac{2k^d}{d(d-2)} \partial_t Z_k - p_1^2 k^{d-2} \left[\frac{d}{(d-1)} Z_k + \frac{d \partial_t Z_k}{(d-1)(d-2)} \right] \right] \right. \\ & \left. + \delta_{d,4} \left[\frac{1}{2} \left[\frac{1}{2} \partial_t Z_k + 2Z_k \right] k^4 - p_1^2 k^2 \left[\frac{1}{2} \partial_t Z_k + Z_k \right] \right] \right\} + \text{sym}\{1, 2, \dots, d\}. \end{aligned} \quad (\text{B.13})$$

We sum (B.2) and (B.13) and write at rank $d = 4$,

$$\begin{aligned} (I^g)_{W;d=4} \simeq & -\frac{\lambda_k}{(Z_k k^2 + \mu_k)^2} \int d\mathbf{P} |\varphi_{p_1 p_2 \dots p_4}|^2 \delta\left(\sum_s p_s\right) \left\{ \right. \\ & \left. \frac{2\pi}{l^4 \sqrt{3}} \left[2 \left[\frac{1}{2} \partial_t Z_k + 2Z_k \right] k^4 - k^2 \left[\frac{1}{2} \partial_t Z_k + Z_k \right] \left(\sum_s p_s^2 \right) \right] + \frac{4}{l^2} \left[k^2 (2 + \partial_t) Z_k - \partial_t Z_k \left(\sum_{s=1}^d p_s^2 \right) \right] \right\} \end{aligned} \quad (\text{B.14})$$

and at rank $d > 4$, summing (B.2) and (B.13) gives

$$(I^g)_W \simeq \frac{\lambda_k}{(Z_k k^2 + \mu_k)^2} \int_{D^*} d\mathbf{p} |\varphi_{p_1 p_2 \dots p_d}|^2 \delta\left(\sum_s p_s\right) \left\{ \begin{aligned} & -\frac{1}{l^d} \frac{\Omega_{d-2}}{\sqrt{d-1}} \left\{ dk^d \left[\frac{(2 + \partial_t) Z_k}{d-2} - \frac{\partial_t Z_k}{d} \right] - k^{d-2} \left[\frac{d \partial_t Z_k}{(d-1)(d-2)} + \frac{d Z_k}{d-1} \right] \left(\sum_s p_s^2 \right) \right\} \\ & - \frac{d}{l^2} \left[k^2 (2 + \partial_t) Z_k - \partial_t Z_k \left(\sum_{s=1}^d p_s^2 \right) \right] \end{aligned} \right\}. \quad (\text{B.15})$$

Hence, we write the β -functions for the couplings μ_k and Z_k at rank $d = 4$, as

$$\begin{aligned} \beta_{d=4}(Z_k) &= \frac{\lambda_k}{(Z_k k^2 + \mu_k)^2} \left\{ \frac{2\pi}{\sqrt{3}} \left[\frac{1}{2} \partial_t Z_k + Z_k \right] \frac{k^2}{l^4} + \partial_t Z_k \frac{4}{l^2} \right\}; \\ \beta_{d=4}(\mu_k) &= -\frac{4\lambda_k}{(Z_k k^2 + \mu_k)^2} \left\{ \frac{\pi}{\sqrt{3}} \left[\frac{1}{2} \partial_t Z_k + 2Z_k \right] \frac{k^4}{l^4} + (2 + \partial_t) Z_k \frac{k^2}{l^2} \right\}, \end{aligned} \quad (\text{B.16})$$

and for $d > 4$, as

$$\begin{aligned} \beta_{d>4}(Z_k) &= \frac{d\lambda_k}{(Z_k k^2 + \mu_k)^2} \left\{ \partial_t Z_k \left[\frac{\pi^{\frac{d-2}{2}}}{(d-1)^{\frac{3}{2}} \Gamma_E\left(\frac{d}{2}\right)} \frac{k^{d-2}}{l^d} + \frac{1}{l^2} \right] + \frac{2\pi^{\frac{d-2}{2}} Z_k}{(d-1)^{\frac{3}{2}} \Gamma_E\left(\frac{d-2}{2}\right)} \frac{k^{d-2}}{l^d} \right\}; \\ \beta_{d>4}(\mu_k) &= -\frac{d\lambda_k}{(Z_k k^2 + \mu_k)^2} \left\{ \partial_t Z_k \left[\frac{k^d}{l^d} \frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1} \Gamma_E\left(\frac{d+2}{2}\right)} + \frac{k^2}{l^2} \right] + 2Z_k \left[\frac{k^d}{l^d} \frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1} \Gamma_E\left(\frac{d}{2}\right)} + \frac{k^2}{l^2} \right] \right\}. \end{aligned} \quad (\text{B.17})$$

We note that setting $d = 3$ in (B.17), we recovers (B.7). We can therefore extend the last formula to $d = 3$, and will denote them $\beta_{d \neq 4}(Z_k)$ and $\beta_{d \neq 4}(\mu_k)$. The case $d = 4$ must be distinguished from the rest of the ranks, because we observe that $\beta_{d=4}(Z_k)$ is not the evaluation of $\beta_{d \neq 4}(Z_k)$ at $d = 4$. Note that the mass equation can be however recovered from $\beta_{d \neq 4}(\mu_k)$ at $d = 4$.

2. φ^4 -terms

The next order of the truncation made on the Wetterich equation, i.e. $(II^g)_W = \text{Tr}[\partial_t R_k \cdot (P_k)^{-1} \cdot F_k \cdot (P_k)^{-1} \cdot F_k \cdot (P_k)^{-1}]$, provides the β -function for the coupling λ_k . Introducing the notation $\hat{\varphi}_{\mathbf{p}} = \varphi_{\mathbf{p}} \delta(\sum p)$ for the gauge invariant field, we write:

$$\begin{aligned} (II^g)_W &= \lambda_k^2 \int_{D^* \times \mathbb{R}^2} d\mathbf{p} d\mathbf{r} \frac{\Theta(k^2 - \sum_s p_s^2) [\partial_t Z_k (k^2 - \sum_s p_s^2) + 2k^2 Z_k] \delta(\sum p)}{(Z_k k^2 + \mu_k)^2 [Z_k \sum_s r_s^2 + \mu_k + \Theta(k^2 - \sum_s r_s^2) Z_k (k^2 - \sum_s r_s^2)] \delta(\sum r) \delta^2(\sum p)} \\ &\times \left[\int_{\mathcal{D}^{d-1}} dm_2 \dots dm_d \hat{\varphi}_{r_1 m_2 \dots m_d} \hat{\varphi}_{p_1 m_2 \dots m_d} \delta(\sum p) \delta(r_1 + p_2 + \dots + p_d) \prod_{i=2}^d \delta(p_i - r_i) \right. \\ &+ \int_{\mathcal{D}} dm_1 \hat{\varphi}_{m_1 r_2 \dots r_d} \hat{\varphi}_{m_1 p_2 \dots p_d} \delta(\sum p) \delta(p_1 + r_2 + \dots + r_d) \delta(p_1 - r_1) + \text{sym}\{1, 2, \dots, d\} \left. \right] \\ &\times \left[\int_{\mathcal{D} \times d-1} dn_2 \dots dn_d \hat{\varphi}_{p_1 n_2 \dots n_d} \hat{\varphi}_{r_1 n_2 \dots n_d} \delta(\sum r) \delta(p_1 + r_2 + \dots + r_d) \prod_{i=2}^d \delta(r_i - p_i) \right. \\ &+ \int_{\mathcal{D}} dn_1 \hat{\varphi}_{n_1 p_2 \dots p_d} \hat{\varphi}_{n_1 r_2 \dots r_d} \delta(\sum r) \delta(r_1 + p_2 + \dots + p_d) \delta(r_1 - p_1) + \text{sym}\{1, 2, \dots, d\} \left. \right], \end{aligned} \quad (\text{B.18})$$

where the redundant Θ -functions are set to 1. The combinatorics of the present model is the same studied in the previous appendix, we therefore proceed in the same way by collecting different types of coloured contributions. We first discuss the contribution obtained by the product of colour 1-1:

$$\begin{aligned} (II^g)_W|_{1,1} &= \lambda_k^2 \int_{D^* \times \mathbb{R}^2} d\mathbf{p} d\mathbf{r} \frac{\Theta(k^2 - \sum_s p_s^2) [\partial_t Z_k (k^2 - \sum_s p_s^2) + 2k^2 Z_k]}{(Z_k k^2 + \mu_k)^2 [Z_k \sum_s r_s^2 + \mu_k + \Theta(k^2 - \sum_s r_s^2) Z_k (k^2 - \sum_s r_s^2)]} \\ &\times \left[\int_{\mathcal{D} \times \mathbb{R}^2} dm_1 dn_1 \hat{\varphi}_{m_1 r_2 \dots r_d} \hat{\varphi}_{m_1 p_2 \dots p_d} \hat{\varphi}_{n_1 p_2 \dots p_d} \hat{\varphi}_{n_1 r_2 \dots r_d} \delta(p_1 + r_2 + \dots + r_d) \delta(r_1 + p_2 + \dots + p_d) \delta^2(r_1 - p_1) \right] \end{aligned}$$

$$\begin{aligned}
& + \int_{\mathcal{D} \times 2d-2} dm_2 \dots dm_d dn_2 \dots dn_d \hat{\varphi}_{r_1 m_2 \dots m_d} \hat{\tilde{\varphi}}_{p_1 m_2 \dots m_d} \hat{\varphi}_{p_1 n_2 \dots n_d} \hat{\tilde{\varphi}}_{r_1 n_2 \dots n_d} \\
& \times \delta(r_1 + p_2 + \dots + p_d) \delta(p_1 + r_2 + \dots + r_d) \prod_{i=2}^d \delta^2(r_i - p_i) \\
& + \text{disconnected} \Big], \tag{B.19}
\end{aligned}$$

where the terms denoted by “disconnected” describe disconnected interactions which we discard. Integrating over r_i , in the delta functions which are not convoluted with the fields, and replacing the redundant δ by $1/l$, one gets:

$$\begin{aligned}
& (II^g)_W|_{1,1} \simeq \lambda_k^2 \int_{D^* \times 2} dm_1 dp_2 \dots dp_d dn_1 dr_2 \dots dr_d \frac{\hat{\varphi}_{m_1 r_2 \dots r_d} \hat{\tilde{\varphi}}_{m_1 p_2 \dots p_d} \hat{\varphi}_{n_1 p_2 \dots p_d} \hat{\tilde{\varphi}}_{n_1 r_2 \dots r_d}}{(Z_k k^2 + \mu_k)^2} \\
& \times \frac{1}{l} \int_{\mathcal{D}} dp_1 \frac{\Theta(k^2 - \Sigma_s p_s^2) [\partial_t Z_k (k^2 - \Sigma_s p_s^2) + 2k^2 Z_k]}{Z_k (p_1^2 + \Sigma_{i=2}^d r_i^2) + \mu_k + \Theta[k^2 - p_1^2 - \Sigma_{i=2}^d r_i^2] Z_k [k^2 - p_1^2 - \Sigma_{i=2}^d r_i^2]} \delta(\Sigma p) \delta(p_1 + \Sigma_{i=2}^d r_i) \\
& + \lambda_k^2 \int_{D^* \times 2} dp_1 dm_2 \dots dm_d dr_1 dn_2 \dots dn_d \frac{\hat{\varphi}_{r_1 m_2 \dots m_d} \hat{\tilde{\varphi}}_{p_1 m_2 \dots m_d} \hat{\varphi}_{p_1 n_2 \dots n_d} \hat{\tilde{\varphi}}_{r_1 n_2 \dots n_d}}{(Z_k k^2 + \mu_k)^2} \\
& \times \frac{1}{l^{d-1}} \int_{\mathcal{D} \times (d-1)} dp_2 \dots dp_d \frac{\Theta(k^2 - \Sigma_s p_s^2) [\partial_t Z_k (k^2 - \Sigma_s p_s^2) + 2k^2 Z_k]}{Z_k (r_1^2 + \Sigma_{i=2}^d p_i^2) + \mu_k + \Theta[k^2 - r_1^2 - \Sigma_{i=2}^d p_i^2] Z_k [k^2 - r_1^2 - \Sigma_{i=2}^d p_i^2]} \\
& \times \delta(\Sigma p) \delta(r_1 + p_2 + \dots + p_d). \tag{B.20}
\end{aligned}$$

Once again, the case $d = 3$ requires a special care during the evaluation of the above integrals. For $d = 3$, we have by direct evaluation:

$$\begin{aligned}
& (II^g)_{W;d=3}|_{1,1} \simeq \lambda_k^2 \int_{D^* \times 2} dm_1 dp_2 dp_3 dn_1 dr_2 dr_3 \frac{\hat{\varphi}_{m_1 r_2 r_3} \hat{\tilde{\varphi}}_{m_1 p_2 p_3} \hat{\varphi}_{n_1 p_2 p_3} \hat{\tilde{\varphi}}_{n_1 r_2 r_3}}{(Z_k k^2 + \mu_k)^2} \\
& \times \frac{1}{l} \int_{\mathcal{D}} dp_1 \frac{\Theta(k^2 - \Sigma_s p_s^2) [\partial_t Z_k (k^2 - \Sigma_s p_s^2) + 2k^2 Z_k]}{Z_k (p_1^2 + \Sigma_{i=2}^3 r_i^2) + \mu_k + \Theta[k^2 - p_1^2 - \Sigma_{i=2}^3 r_i^2] Z_k [k^2 - p_1^2 - \Sigma_{i=2}^3 r_i^2]} \delta(\Sigma p) \delta(p_1 + r_2 + r_3) \\
& + \lambda_k^2 \int_{D^* \times 2} dp_1 dm_2 dm_3 dr_1 dn_2 dn_3 \frac{\hat{\varphi}_{r_1 m_2 m_3} \hat{\tilde{\varphi}}_{p_1 m_2 m_3} \hat{\varphi}_{p_1 n_2 n_3} \hat{\tilde{\varphi}}_{r_1 n_2 n_3}}{(Z_k k^2 + \mu_k)^2} \\
& \times \frac{1}{l^2} \int_{\mathcal{D} \times 2} dp_2 dp_3 \frac{\Theta(k^2 - \Sigma_s p_s^2) [\partial_t Z_k (k^2 - \Sigma_s p_s^2) + 2k^2 Z_k]}{Z_k (r_1^2 + \Sigma_{i=2}^3 p_i^2) + \mu_k + \Theta[k^2 - r_1^2 - \Sigma_{i=2}^3 p_i^2] Z_k [k^2 - r_1^2 - \Sigma_{i=2}^3 p_i^2]} \delta(\Sigma p) \delta(r_1 + p_2 + p_3). \tag{B.21}
\end{aligned}$$

We integrate over p_1 the first term and over p_3 the second term, replace redundant deltas by appropriate factors $1/l$ and then put to 0 all momentum variables involved in the field convolutions, to get

$$\begin{aligned}
& (II^g)_{W;d=3}|_{1,1} \simeq \frac{\lambda_k^2}{(Z_k k^2 + \mu_k)^3} \frac{k^2 (2 + \partial_t) Z_k}{l^2} \mathcal{V}_1 \\
& + \frac{\lambda_k^2}{(Z_k k^2 + \mu_k)^3} \frac{1}{l^3} \int_{-\sqrt{k^2/2}}^{\sqrt{k^2/2}} dr [\partial_t Z_k (k^2 - 2r^2) + 2k^2 Z_k] \mathcal{V}_1 \\
& \simeq \frac{\lambda_k^2}{(Z_k k^2 + \mu_k)^3} \left[\frac{k^2 (2 + \partial_t) Z_k}{l^2} + \frac{k^3}{l^3} \left[\sqrt{2} (\partial_t + 2) Z_k - \frac{\sqrt{2}}{3} \partial_t Z_k \right] \right] \mathcal{V}_1 \\
& \simeq \frac{\lambda_k^2}{(Z_k k^2 + \mu_k)^3} \left[\left[\frac{k^2}{l^2} + \frac{2\sqrt{2}}{3} \frac{k^3}{l^3} \right] \partial_t Z_k + 2 \left[\frac{k^2}{l^2} + \sqrt{2} \frac{k^3}{l^3} \right] Z_k \right] \mathcal{V}_1. \tag{B.22}
\end{aligned}$$

At rank $d > 3$, using again the spherical coordinates (R, Ω_{d-1}) , and taking the continuum limit, we write:

$$\begin{aligned}
& (II^g)_{W;d>3}|_{1,1} \simeq \lambda_k^2 \int dm_1 dp_2 \dots dp_d dn_1 dr_2 \dots dr_d \frac{\hat{\varphi}_{m_1 r_2 \dots r_d} \hat{\tilde{\varphi}}_{m_1 p_2 \dots p_d} \hat{\varphi}_{n_1 p_2 \dots p_d} \hat{\tilde{\varphi}}_{n_1 r_2 \dots r_d}}{l (Z_k k^2 + \mu_k)^2} \\
& \times \int dp_1 \frac{\Theta(k^2 - \Sigma_s p_s^2) [\partial_t Z_k (k^2 - \Sigma_s p_s^2) + 2k^2 Z_k]}{Z_k (p_1^2 + \Sigma_{i=2}^d r_i^2) + \mu_k + \Theta[k^2 - p_1^2 - \Sigma_{i=2}^d r_i^2] Z_k [k^2 - p_1^2 - \Sigma_{i=2}^d r_i^2]} \delta(\Sigma p) \delta(p_1 + \Sigma_{i=2}^d r_i) \\
& + \lambda_k^2 \int dp_1 dm_2 \dots dm_d dr_1 dn_2 \dots dn_d \frac{\hat{\varphi}_{r_1 m_2 \dots m_d} \hat{\tilde{\varphi}}_{p_1 m_2 \dots m_d} \hat{\varphi}_{p_1 n_2 \dots n_d} \hat{\tilde{\varphi}}_{r_1 n_2 \dots n_d}}{l^{d-1} (Z_k k^2 + \mu_k)^2} \\
& \int dR \int d\Omega_{d-1} \frac{R^{d-2} \theta(k^2 - p_1^2 - R^2) [\partial_t Z_k (k^2 - p_1^2 - R^2) + 2k^2 Z_k]}{Z_k (r_1^2 + R^2) + \mu_k + \theta(k^2 - r_1^2 - R^2) Z_k [k^2 - r_1^2 - R^2]}
\end{aligned}$$

$$\begin{aligned}
& \times \delta(p_1 + R\sqrt{d-1} \cos \vartheta) \delta(r_1 + R\sqrt{d-1} \cos \vartheta) \\
& = \lambda_k^2 \int dm_1 dp_2 \dots dp_d dn_1 dr_2 \dots dr_d \frac{\hat{\varphi}_{m_1 r_2 \dots r_d} \hat{\varphi}_{m_1 p_2 \dots p_d} \hat{\varphi}_{n_1 p_2 \dots p_d} \hat{\varphi}_{n_1 r_2 \dots r_d}}{l(Z_k k^2 + \mu_k)^2} \\
& \times \int dp_1 \frac{\theta(k^2 - \Sigma_s p_s^2) [\partial_t Z_k (k^2 - \Sigma_s p_s^2) + 2k^2 Z_k]}{Z_k (p_1^2 + \Sigma_{i=2}^d r_i^2) + \mu_k + \theta[k^2 - p_1^2 - \Sigma_{i=2}^d r_i^2] Z_k [k^2 - p_1^2 - \Sigma_{i=2}^d r_i^2]} \delta(\Sigma p) \delta(p_1 + \Sigma_{i=2}^d r_i) \\
& + \lambda_k^2 \int dp_1 dm_2 \dots dm_d dr_1 dn_2 \dots dn_d \frac{\hat{\varphi}_{r_1 m_2 \dots m_d} \hat{\varphi}_{p_1 m_2 \dots m_d} \hat{\varphi}_{p_1 n_2 \dots n_d} \hat{\varphi}_{r_1 n_2 \dots n_d}}{l^{d-1} (Z_k k^2 + \mu_k)^2} \\
& \times \int dR \int d\Omega_{d-2} \int_0^\pi \frac{R^{d-2}}{R^2(d-1)} d\vartheta \sin^{d-3} \vartheta \delta\left(\frac{p_1}{R\sqrt{d-1}} + \cos \vartheta\right) \delta\left(\frac{r_1}{R\sqrt{d-1}} + \cos \vartheta\right) \\
& \times \frac{\theta(k^2 - p_1^2 - R^2) [\partial_t Z_k (k^2 - p_1^2 - R^2) + 2k^2 Z_k]}{Z_k (r_1^2 + R^2) + \mu_k + \theta(k^2 - r_1^2 - R^2) Z_k (k^2 - r_1^2 - R^2)} \\
& = \lambda_k^2 \int dm_1 dp_2 \dots dp_d dn_1 dr_2 \dots dr_d \frac{\hat{\varphi}_{m_1 r_2 \dots r_d} \hat{\varphi}_{m_1 p_2 \dots p_d} \hat{\varphi}_{n_1 p_2 \dots p_d} \hat{\varphi}_{n_1 r_2 \dots r_d}}{l(Z_k k^2 + \mu_k)^2} \\
& \times \frac{\theta\left[k^2 - 2\left(\Sigma_{i=2}^d p_i^2 + \Sigma_{1 < i < j} p_i p_j\right)\right] \left\{ \partial_t Z_k \left[k^2 - 2\left(\Sigma_{i=2}^d p_i^2 + \Sigma_{1 < i < j} p_i p_j\right)\right] + 2k^2 Z_k \right\} \delta(\Sigma_{i=2}^d (r_i - p_i))}{Z_k [\Sigma_{i=2}^d r_i^2 + (\Sigma_{i=2}^d p_i)^2] + \mu_k + \theta[k^2 - \Sigma_{i=2}^d r_i^2 - (\Sigma_{i=2}^d p_i)^2] Z_k [k^2 - \Sigma_{i=2}^d r_i^2 - (\Sigma_{i=2}^d p_i)^2]} \\
& + \lambda_k^2 \Omega_{d-2} \int dp_1 dm_2 \dots dm_d dr_1 dn_2 \dots dn_d \frac{\hat{\varphi}_{r_1 m_2 \dots m_d} \hat{\varphi}_{p_1 m_2 \dots m_d} \hat{\varphi}_{p_1 n_2 \dots n_d} \hat{\varphi}_{r_1 n_2 \dots n_d}}{l^{d-1} (Z_k k^2 + \mu_k)^2} \\
& \times \int dR \left[1 - \frac{r_1^2}{R^2(d-1)}\right]^{\frac{d-4}{2}} \frac{R^{d-3}}{\sqrt{d-1}} \delta(p_1 - r_1) \frac{\theta(k^2 - p_1^2 - R^2) [\partial_t Z_k (k^2 - p_1^2 - R^2) + 2k^2 Z_k]}{Z_k (r_1^2 + R^2) + \mu_k + \theta(k^2 - r_1^2 - R^2) Z_k (k^2 - r_1^2 - R^2)}. \tag{B.23}
\end{aligned}$$

Considering that all interaction terms which explicitly depend on the momenta involved in their fields fall out of our truncation, considering also that the deltas $\delta(p_1 - r_1)$ and $\delta(\Sigma_{i=2}^d (r_i - p_i))$ turn out to be redundant with the gauge invariance conditions, we can then set to zero the labels p_i and r_i appearing in the integrals, coefficients of the gauge projected fields, and get:

$$\begin{aligned}
& (II^g)_{W; d>3|_{1,1}} \simeq \frac{\lambda_k^2 k^2 (2 + \partial_t) Z_k}{l^2 (Z_k k^2 + \mu_k)^3} \int_{D^{* \times 2}} dm_1 dp_2 \dots dp_d dn_1 dr_2 \dots dr_d \hat{\varphi}_{m_1 r_2 \dots r_d} \hat{\varphi}_{m_1 p_2 \dots p_d} \hat{\varphi}_{n_1 p_2 \dots p_d} \hat{\varphi}_{n_1 r_2 \dots r_d} \\
& + \frac{2\pi^{\frac{d-2}{2}} \lambda_k^2}{(Z_k k^2 + \mu_k)^3 \Gamma_E\left(\frac{d-2}{2}\right) \sqrt{d-1}} \frac{k^d}{l^d} \left[\frac{(2 + \partial_t) Z_k}{d-2} - \frac{\partial_t Z_k}{d} \right] \\
& \times \int_{D^{* \times 2}} dp_1 dm_2 \dots dm_d dr_1 dn_2 \dots dn_d \hat{\varphi}_{r_1 m_2 \dots m_d} \hat{\varphi}_{p_1 m_2 \dots m_d} \hat{\varphi}_{p_1 n_2 \dots n_d} \hat{\varphi}_{r_1 n_2 \dots n_d} \\
& = \frac{\lambda_k^2}{(Z_k k^2 + \mu_k)^3} \left\{ (2 + \partial_t) Z_k \left[\frac{\pi^{\frac{d-2}{2}}}{\Gamma_E\left(\frac{d}{2}\right) \sqrt{d-1}} \frac{k^d}{l^d} + \frac{k^2}{l^2} \right] - \frac{2\pi^{\frac{d-2}{2}} \partial_t Z_k}{d \sqrt{d-1} \Gamma_E\left(\frac{d-2}{2}\right)} \frac{k^d}{l^d} \right\} \mathcal{V}_1, \tag{B.24}
\end{aligned}$$

where we used for the coloured vertex the same notation introduced in section A 2. We note that setting $d = 3$ in the last result leads us to (B.22). Then, we prolong $(II^g)_W$ to $d \geq 3$.

Inspecting the 2-colour cross terms, we focus on the product of terms 1-2. Discarding the disconnected interactions and the terms which fall out of the chosen truncation, while paying a special care on the case $d = 3$, one has:

$$\begin{aligned}
& (II^g)_W|_{1,2} \simeq \lambda_k^2 \int_{D^{* \times 2}} d\mathbf{p} d\mathbf{r} \frac{\Theta(k^2 - \Sigma_s p_s^2) [\partial_t Z_k (k^2 - \Sigma_s p_s^2) + 2k^2 Z_k]}{(Z_k k^2 + \mu_k)^2 [Z_k \Sigma_s r_s^2 + \mu_k + \Theta(k^2 - \Sigma_s r_s^2) Z_k (k^2 - \Sigma_s r_s^2)]} \\
& \times \left[\delta_{d,3} \int_{D^2} dm_1 dn_2 \hat{\varphi}_{m_1 r_2 r_3} \hat{\varphi}_{m_1 p_2 p_3} \hat{\varphi}_{p_1 n_2 p_3} \hat{\varphi}_{r_1 n_2 r_3} \delta(p_1 + r_2 + p_3) \delta(p_1 + r_2 + r_3) \delta(r_1 - p_1) \delta(p_2 - r_2) \right. \\
& + \int_{D^*} dm_1 dn_1 dn_3 \dots dn_d \hat{\varphi}_{m_1 r_2 \dots r_d} \hat{\varphi}_{m_1 p_2 \dots p_d} \hat{\varphi}_{n_1 p_2 n_3 \dots n_d} \hat{\varphi}_{n_1 r_2 n_3 \dots n_d} \\
& \times \delta(p_1 + r_2 + \dots + r_d) \delta(r_1 + p_2 + r_3 + \dots + r_d) \delta^2(r_1 - p_1) \prod_{i=3}^d \delta(p_i - r_i) \\
& + \int_{D^*} dn_2 dm_2 \dots dm_d \hat{\varphi}_{r_1 m_2 \dots m_d} \hat{\varphi}_{p_1 m_2 \dots m_d} \hat{\varphi}_{p_1 n_2 p_3 \dots p_d} \hat{\varphi}_{r_1 n_2 r_3 \dots p_d} \\
& \left. \times \delta(r_1 + p_2 + \dots + p_d) \delta(p_1 + r_2 + p_3 + \dots + p_d) \delta^2(p_2 - r_2) \prod_{i=3}^d \delta(p_i - r_i) \right]
\end{aligned}$$

$$\begin{aligned}
&\simeq \frac{\lambda_k^2}{(Z_k k^2 + \mu_k)^2} \left\{ \delta_{d,3} \int_{D^2} dm_1 dn_2 dr_3 dp_1 dp_2 dp_3 \hat{\varphi}_{m_1 p_2 r_3} \hat{\varphi}_{m_1 p_2 p_3} \hat{\varphi}_{p_1 n_2 p_3} \hat{\varphi}_{p_1 n_2 r_3} \delta(p_1 + p_2 + p_3) \delta(p_1 + p_2 + r_3) \right. \\
&\times \frac{\Theta(k^2 - \Sigma_s p_s^2) [\partial_t Z_k (k^2 - \Sigma_s p_s^2) + 2k^2 Z_k]}{Z_k (p_1^2 + p_2^2 + r_3^2) + \mu_k + \Theta[k^2 - (p_1^2 + p_2^2 + r_3^2)] Z_k [k^2 - (p_1^2 + p_2^2 + r_3^2)]} \\
&+ \frac{1}{l} \int_{D^{* \times 2}} dm_1 dp_2 \dots dp_d dn_1 dr_2 dn_3 \dots dn_d \hat{\varphi}_{m_1 r_2 p_3 \dots p_d} \hat{\varphi}_{m_1 p_2 \dots p_d} \hat{\varphi}_{n_1 p_2 n_3 \dots n_d} \hat{\varphi}_{n_1 r_2 n_3 \dots n_d} \\
&\times \int_{\mathcal{D}} dp_1 \delta(p_1 + r_2 + p_3 + \dots + p_d) \delta(\Sigma p) \\
&\times \frac{\Theta(k^2 - \Sigma_s p_s^2) [\partial_t Z_k (k^2 - \Sigma_s p_s^2) + 2k^2 Z_k]}{Z_k (p_1^2 + r_2^2 + \Sigma_{i=3}^d p_i^2) + \mu_k + \Theta[k^2 - (p_1^2 + r_2^2 + \Sigma_{i=3}^d p_i^2)] Z_k [k^2 - (p_1^2 + r_2^2 + \Sigma_{i=3}^d p_i^2)]} \\
&+ \frac{1}{l} \int_{D^{* \times 2}} dr_1 dm_2 \dots dm_d dp_1 dn_2 dp_3 \dots dp_d \hat{\varphi}_{r_1 m_2 \dots m_d} \hat{\varphi}_{p_1 m_2 \dots m_d} \hat{\varphi}_{p_1 n_2 p_3 \dots p_d} \hat{\varphi}_{r_1 n_2 p_3 \dots p_d} \\
&\times \int_{\mathcal{D}} dp_2 \delta(r_1 + p_2 + \dots + p_d) \delta(\Sigma p) \\
&\times \frac{\Theta(k^2 - \Sigma_s p_s^2) [\partial_t Z_k (k^2 - \Sigma_s p_s^2) + 2k^2 Z_k]}{Z_k (r_1^2 + p_2^2 + \dots + p_d^2) + \mu_k + \Theta[k^2 - (r_1^2 + p_2^2 + \dots + p_d^2)] Z_k [k^2 - (r_1^2 + p_2^2 + \dots + p_d^2)]} \Big\}. \tag{B.25}
\end{aligned}$$

Performing the integral over p_1 and p_2 in the last two terms and evaluating at the 0-momentum we find:

$$\begin{aligned}
(II^g)_W|_{1,2} &\simeq \delta_{d,3} \frac{\lambda_k^2}{(Z_k k^2 + \mu_k)^3} \frac{k^2}{l^2} (2 + \partial_t) Z_k \mathcal{V}_3 + \frac{\lambda_k^2}{(Z_k k^2 + \mu_k)^3} \frac{k^2}{l^2} (2 + \partial_t) Z_k [\mathcal{V}_2 + \mathcal{V}_1] \\
&\simeq \frac{\lambda_k^2}{(Z_k k^2 + \mu_k)^3} \frac{k^2}{l^2} (2 + \partial_t) Z_k [\delta_{d,3} \mathcal{V}_3 + \mathcal{V}_2 + \mathcal{V}_1] \tag{B.26}
\end{aligned}$$

The combinatorics of the φ^4 is the same with or without the presence of (gauge) constraints, the contribution to the coefficients coming from the color symmetry is the same as for the previous model. Collecting all contributions, $(II^g)_W|_{i,i}$ (B.24), $i = 1, \dots, d$, and $(II^g)_W|_{i,j}$ (B.26), $i < j$, $i, j = 1, \dots, d$, the β -function for λ_k , in any rank d , expresses as

$$\beta(\lambda_k) = \frac{2\lambda_k^2}{(Z_k k^2 + \mu_k)^3} \left\{ \partial_t Z_k \left[\frac{2\pi^{\frac{d-2}{2}}}{d\sqrt{d-1}\Gamma\left(\frac{d}{2}\right)} \frac{k^d}{l^d} + (2d-1) \frac{k^2}{l^2} \right] + 2Z_k \left[\frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1}\Gamma\left(\frac{d}{2}\right)} \frac{k^d}{l^d} + (2d-1) \frac{k^2}{l^2} \right] \right\}. \tag{B.27}$$

Dimensionful β -functions. Let us collect all β -functions. At rank $d \neq 4$, we gather (B.17) and (B.27) for the complete system of β -functions for the gauge invariant TGFT model which expresses as:

$$\left\{ \begin{aligned} \beta_{d>4}(Z_k) &= \frac{d\lambda_k}{(Z_k k^2 + \mu_k)^2} \left\{ \partial_t Z_k \left[\frac{\pi^{\frac{d-2}{2}}}{(d-1)^{\frac{3}{2}} \Gamma_E\left(\frac{d}{2}\right)} \frac{k^{d-2}}{l^d} + \frac{1}{l^2} \right] + \frac{2\pi^{\frac{d-2}{2}} Z_k}{(d-1)^{\frac{3}{2}} \Gamma_E\left(\frac{d-2}{2}\right)} \frac{k^{d-2}}{l^d} \right\} \\ \beta_{d \neq 4}(\mu_k) &= -\frac{d\lambda_k}{(Z_k k^2 + \mu_k)^2} \left\{ \partial_t Z_k \left[\frac{k^d}{l^d} \frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1}\Gamma_E\left(\frac{d+2}{2}\right)} + \frac{k^2}{l^2} \right] + 2Z_k \left[\frac{k^d}{l^d} \frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1}\Gamma_E\left(\frac{d}{2}\right)} + \frac{k^2}{l^2} \right] \right\} \\ \beta_{d \neq 4}(\lambda_k) &= \frac{2\lambda_k^2}{(Z_k k^2 + \mu_k)^3} \left\{ \partial_t Z_k \left[\frac{2\pi^{\frac{d-2}{2}}}{d\sqrt{d-1}\Gamma\left(\frac{d}{2}\right)} \frac{k^d}{l^d} + (2d-1+2\delta_{d,3}) \frac{k^2}{l^2} \right] + 2Z_k \left[\frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1}\Gamma\left(\frac{d}{2}\right)} \frac{k^d}{l^d} + (2d-1+2\delta_{d,3}) \frac{k^2}{l^2} \right] \right\} \end{aligned} \right\} \tag{B.28}$$

which is reported in (93) in section IV C and at $d = 4$, we obtain the expression

$$\left\{ \begin{aligned} \beta_{d=4}(Z_k) &= \frac{\lambda_k}{(Z_k k^2 + \mu_k)^2} \left\{ \frac{2\pi}{\sqrt{3}} \left[\frac{1}{2} \partial_t Z_k + Z_k \right] \frac{k^2}{l^4} + \partial_t Z_k \frac{4}{l^2} \right\} \\ \beta_{d=4}(\mu_k) &= -\frac{4\lambda_k}{(Z_k k^2 + \mu_k)^2} \left\{ \frac{\pi}{\sqrt{3}} \left[\frac{1}{2} \partial_t Z_k + 2Z_k \right] \frac{k^4}{l^4} + (2 + \partial_t) Z_k \frac{k^2}{l^2} \right\} \\ \beta_{d=4}(\lambda_k) &= \frac{2\lambda_k^2}{(Z_k k^2 + \mu_k)^3} \left\{ \partial_t Z_k \left[\frac{2\pi}{4\sqrt{3}} \frac{k^4}{l^4} + 7 \frac{k^2}{l^2} \right] + 2Z_k \left[\frac{\pi}{\sqrt{3}} \frac{k^4}{l^4} + 7 \frac{k^2}{l^2} \right] \right\} \end{aligned} \right\} \tag{B.29}$$

as reported in (94).

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